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5D supergravity and projective superspace

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Abstract: This paper is a companion to our earlier work [1] in which the projective superspace formulation for matter-coupled simple supergravity in five dimensions was presented. For the minimal multiplet of 5D $\mathcal{N}=1$ supergravity introduced by Howe in 1981, we give a complete solution of the Bianchi identities. The geometry of curved superspace is shown to allow the existence of a large family of off-shell supermultiplets that can be used to describe supersymmetric matter, including vector multiplets and hypermultiplets. We formulate a manifestly locally supersymmetric action principle. Its natural property turns out to be the invariance under so-called projective transformations of the auxiliary isotwistor variables. We then demonstrate that the projective invariance allows one to uniquely restore the action functional in a Wess-Zumino gauge. The latter action is well-suited for reducing the supergravity-matter systems to components.

Keywords: Superspaces, Extended Supersymmetry, Supergravity Models.

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Contents

1. Introduction

In our recent paper [1], the projective superspace formulation for matter-coupled simple supergravity in five dimensions was presented. Building on the earlier work of [2], ref. [1] provided the first solution to the important old problem of incorporating supergravity into the projective superspace approach [3, 4]. The latter is known to be a powerful paradigm for constructing off-shell rigid supersymmetric theories with eight supercharges in $D \leq 6$ space-time dimensions, and in particular for the explicit construction of hyperkähler metrics, see, e.g., [5]. In [1], we introduced various supermultiplets to describe matter fields coupled to supergravity, stated the locally supersymmetric action principle in the Wess-Zumino gauge, and constructed several interesting supergravity-matter systems.

The present paper, on one hand, is a companion to [1]. Here we derive those technical details that were stated in [1] without proof. In particular, we show that the requirement of projective invariance allows one to uniquely reconstruct the locally supersymmetric action in the Wess-Zumino gauge. On the other hand, this paper contains an important new result. Specifically, we formulate a manifestly locally supersymmetric action that reduces to that given in [1] upon imposing the Wess-Zumino gauge. This result completes the formal structure of 5D $\mathcal{N} = 1$ superfield supergravity.

Before turning to the technical aspects of this work, we would like to give two general comments. First, five-dimensional $\mathcal{N} = 1$ supergravity[1] [6] and its matter couplings have extensively been studied at the component level, both in on-shell [7−9] and off-shell [10−12] settings. It is thus natural to ask: Are there still good reasons for developing superspace formulations? We believe the answer is "Yes." There are several ways to justify this claim, and the most practical is the following. Unlike the component schemes developed, superspace approaches have the potential to offer a generating formalism to realize most general sigma-model couplings, and hence to construct general quaternionic Kähler manifolds. It is instructive to discuss the situation with hypermultiplets. In the component formulations[2] of [10, 11], one makes use of an off-shell realization for the hypermultiplet with finitely many auxiliary fields and an intrinsic central charge. As is well-known, it is the presence of central charge which makes it impossible to cast general quaternionic Kähler couplings in terms of such off-shell hypermultiplets. On the other hand, the projective superspace approach offers nice off-shell formulations without central charge. Specifically, there are infinitely many off-shell realizations with *finitely many auxiliary fields* for a neutral hypermultiplet (they are the called $O(2n)$ multiplets, where $n = 2, 3\dots$, following the terminology of [13]), and a unique formulation for a charged hypermultiplet with *infinitely many auxiliary fields* (the so-called polar hypermultiplet).

Our second comment concerns the choice made in this paper to use the projective superspace setting to formulate supergravity-matter systems. Why not harmonic superspace [14, 15]? As is known, both approaches can be used to describe supersymmetric theories with eight supercharges in $D \leq 6$ space-time dimensions. There are, however, two major differences between them: (i) the structure of off-shell supermultiplets used; and (ii) the supersymmetric action principle chosen. It is due to these differences that the two approaches are complementary to each other in some respects. From the point of view of supergravity theories with eight supercharges in $D \leq 6$ space-time dimensions, harmonic superspace offers powerful prepotential formulations [16, 17]. On the other hand, as will be shown in this paper, projective superspace is ideal for developing covariant geometric formulations for supergravity-matter systems, similar to the famous Wess-Zumino approach for 4D $\mathcal{N} = 1$ supergravity [18]. The point is that projective superspace is a robust scheme for supersymmetric model-buliding, see, e.g., [19] for the recent construction of hyperkähler metrics on cotangent bundles of Hermitian symmetric spaces.

This paper is organized as follows. In section 2 we provide a complete solution of the Bianchi identities for the superspace geometry corresponding to the minimal 5D $\mathcal{N} = 1$ supergravity multiplet [20]. In section 3 we formulate, following [1], off-shell projective supermultiplets, and then construct a manifestly locally supersymmetric action. Section 4 is devoted to the technicalities of the Wess-Zumino gauge for supergravity. Section 5 demonstrates that the locally supersymmetric action in the Wess-Zumino gauge is uniquely determined from the requirement of projective invariance. Our 5D conventions and useful identities are collected in the appendix.

[1]On historical grounds, 5D simple ($\mathcal{N} = 1$) supersymmetry and supergravity are often labeled $\mathcal{N} = 2$.
[2]Refs. [12] deal with on-shell hypermultiplets only.

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2. Superspace geometry of the minimal supergravity multiplet

In this section we present a complete solution to the Bianchi identities for the constraints on the superspace torsions that were introduced by Howe[3] in 1981 [20] and correspond to the so-called minimal 5D $\mathcal{N}=1$ supergravity multiplet.[4] The results of this section were used in [1] without proof.

Let $z^{\hat{M}} = (x^{\hat{m}}, \theta^{\hat{\mu}}_i)$ be local bosonic (x) and fermionic (θ) coordinates parametrizing a curved five-dimensional $\mathcal{N}=1$ superspace $\mathcal{M}^{5|8}$, where $\hat{m} = 0, 1, \dots, 4$, $\hat{\mu} = 1, \dots, 4$, and $i = \underline{1}, \underline{2}$. The Grassmann variables $\theta^{\hat{\mu}}_i$ are assumed to obey the standard pseudo-Majorana reality condition $(\theta^{\hat{\mu}}_i)^* = \theta^i_{\hat{\mu}} = \varepsilon_{\hat{\mu}\hat{\nu}}\,\varepsilon^{ij}\,\theta^{\hat{\nu}}_j$ (see the appendix for our 5D notation and conventions). Following [20], the tangent-space group is chosen to be $\mathrm{SO}(4,1)\times\mathrm{SU}(2)$, and the superspace covariant derivatives $\mathcal{D}_{\hat{A}} = (\mathcal{D}_{\hat{a}}, \mathcal{D}^i_{\hat{\alpha}}) \equiv (\mathcal{D}_{\hat{a}}, \mathcal{D}_{\underline{\hat{\alpha}}})$ have the form

$$\mathcal{D}_{\hat{A}} = E_{\hat{A}} + \Omega_{\hat{A}} + \Phi_{\hat{A}} + V_{\hat{A}} Z\ . \tag{2.1}$$

Here $E_{\hat{A}} = E_{\hat{A}}{}^{\hat{M}}(z)\,\partial_{\hat{M}}$ is the supervielbein, with $\partial_{\hat{M}} = \partial/\partial z^{\hat{M}}$,

$$\Omega_{\hat{A}} = \frac{1}{2}\,\Omega_{\hat{A}}{}^{\hat{b}\hat{c}}\,M_{\hat{b}\hat{c}} = \Omega_{\hat{A}}{}^{\hat{\beta}\hat{\gamma}}\,M_{\hat{\beta}\hat{\gamma}}\,, \qquad M_{\hat{a}\hat{b}} = -M_{\hat{b}\hat{a}}\,, \quad M_{\hat{\alpha}\hat{\beta}} = M_{\hat{\beta}\hat{\alpha}} \tag{2.2}$$

is the Lorentz connection,

$$\Phi_{\hat{A}} = \Phi_{\hat{A}}{}^{kl}\,J_{kl}\,, \qquad J_{kl} = J_{lk} \tag{2.3}$$

is the SU(2)-connection, and Z the central-charge generator, $[Z, \mathcal{D}_{\hat{A}}] = 0$. The Lorentz generators with vector indices ($M_{\hat{a}\hat{b}}$) and spinor indices ($M_{\hat{\alpha}\hat{\beta}}$) are related to each other by the rule: $M_{\hat{a}\hat{b}} = (\Sigma_{\hat{a}\hat{b}})^{\hat{\alpha}\hat{\beta}} M_{\hat{\alpha}\hat{\beta}}$ (for more details, see the appendix). The generators of $\mathrm{SO}(4,1)\times\mathrm{SU}(2)$ act on the covariant derivatives as follows:[5]

$$[J^{kl}, \mathcal{D}^i_{\hat{\alpha}}] = \varepsilon^{i(k}\mathcal{D}^{l)}_{\hat{\alpha}}\,, \quad [M_{\hat{\alpha}\hat{\beta}}, \mathcal{D}^k_{\hat{\gamma}}] = \varepsilon_{\hat{\gamma}(\hat{\alpha}}\mathcal{D}^k_{\hat{\beta})}\,, \quad [M_{\hat{a}\hat{b}}, \mathcal{D}_{\hat{c}}] = 2\eta_{\hat{c}[\hat{a}}\mathcal{D}_{\hat{b}]}\,, \tag{2.4}$$

where $J^{kl} = \varepsilon^{ki}\varepsilon^{lj}J_{ij}$.

The supergravity gauge group is generated by local transformations of the form

$$\mathcal{D}_{\hat{A}} \to \mathcal{D}'_{\hat{A}} = \mathrm{e}^{K}\,\mathcal{D}_{\hat{A}}\,\mathrm{e}^{-K}\,, \qquad K = K^{\hat{C}}(z)\mathcal{D}_{\hat{C}} + \frac{1}{2}K^{\hat{c}\hat{d}}(z)M_{\hat{c}\hat{d}} + K^{kl}(z)J_{kl} + \tau(z)Z\,, \tag{2.5}$$

with all the gauge parameters being neutral with respect to the central charge Z, obeying natural reality conditions, and otherwise arbitrary. Given a tensor superfield $U(z)$, with its indices suppressed, it transforms as follows:

$$U \to U' = \mathrm{e}^{K}\,U\ . \tag{2.6}$$

[3] The choice of the constraints given in [20] was motivated by the structure of the 5D $\mathcal{N}=1$ supercurrent [21].

[4] This supermultiplet was re-discovered almost twenty years later by Zucker [10] who essentially elaborated the component implications of the superspace formulation given in [20].

[5] The operation of (anti)symmetrization of n indices is defined to involve a factor $(n!)^{-1}$.

The covariant derivatives obey (anti)commutation relations of the general form

$$[\mathcal{D}_{\hat{A}}, \mathcal{D}_{\hat{B}}\} = T_{\hat{A}\hat{B}}{}^{\hat{C}}\mathcal{D}_{\hat{C}} + \frac{1}{2}R_{\hat{A}\hat{B}}{}^{\hat{c}\hat{d}}M_{\hat{c}\hat{d}} + R_{\hat{A}\hat{B}}{}^{kl}J_{kl} + F_{\hat{A}\hat{B}}Z\,, \tag{2.7}$$

where $T_{\hat{A}\hat{B}}{}^{\hat{C}}$ is the torsion, $R_{\hat{A}\hat{B}}{}^{kl}$ and $R_{\hat{A}\hat{B}}{}^{\hat{c}\hat{d}}$ the SU(2)- and SO(4,1)-curvature tensors, respectively, and $F_{\hat{A}\hat{B}}$ the central charge field strength.

The Bianchi identities are:

$$\sum_{(\hat{A}\hat{B}\hat{C})} [\mathcal{D}_{\hat{A}}, [\mathcal{D}_{\hat{B}}, \mathcal{D}_{\hat{C}}\}\} \;=\; 0\,, \tag{2.8}$$

with the graded cyclic sum assumed. The Bianchi identities are equivalent to the following equations on the torsion and curvature tensors:

$$0 = \sum_{(\hat{A}\hat{B}\hat{C})} \Big(R_{\hat{A}\hat{B}\hat{C}}{}^{\hat{D}} - \mathcal{D}_{\hat{A}}T_{\hat{B}\hat{C}}{}^{\hat{D}} + T_{\hat{A}\hat{B}}{}^{\hat{E}}T_{\hat{E}\hat{C}}{}^{\hat{D}} \Big)\,, \tag{2.9a}$$

$$0 = \sum_{(\hat{A}\hat{B}\hat{C})} \Big(\mathcal{D}_{\hat{A}}R_{\hat{B}\hat{C}}{}^{kl} - T_{\hat{A}\hat{B}}{}^{\hat{D}}R_{\hat{D}\hat{C}}{}^{kl} \Big)\,, \quad 0 = \sum_{(\hat{A}\hat{B}\hat{C})} \Big(\mathcal{D}_{\hat{A}}R_{\hat{B}\hat{C}}{}^{\hat{\rho}\hat{\tau}} - T_{\hat{A}\hat{B}}{}^{\hat{D}}R_{\hat{D}\hat{C}}{}^{\hat{\rho}\hat{\tau}} \Big)\,, \tag{2.9b}$$

$$0 = \sum_{(\hat{A}\hat{B}\hat{C})} \Big(\mathcal{D}_{\hat{A}}F_{\hat{B}\hat{C}} - T_{\hat{A}\hat{B}}{}^{\hat{D}}F_{\hat{D}\hat{C}} \Big)\,, \tag{2.9c}$$

where[6]

$$R_{\hat{A}\hat{B}\hat{C}}{}^{\hat{D}} \equiv R_{\hat{A}\hat{B}}{}^{\hat{\rho}\hat{\tau}}(M_{\hat{\rho}\hat{\tau}})_{\hat{C}}{}^{\hat{D}} + R_{\hat{A}\hat{B}}{}^{kl}(J_{kl})_{\hat{C}}{}^{\hat{D}}\,, \tag{2.10a}$$

$$[M_{\hat{\delta}\hat{\rho}}, \mathcal{D}_{\hat{A}}] \equiv (M_{\hat{\delta}\hat{\rho}})_{\hat{A}}{}^{\hat{B}}\mathcal{D}_{\hat{B}}\,, \qquad [J_{kl}, \mathcal{D}_{\hat{A}}] \equiv (J_{kl})_{\hat{A}}{}^{\hat{B}}\mathcal{D}_{\hat{B}}\,, \tag{2.10b}$$

$$(M_{\hat{\rho}\hat{\tau}})_{\underline{\hat{\alpha}}}{}^{\underline{\hat{\beta}}} = \delta^i_j\varepsilon_{\hat{\alpha}(\hat{\rho}}\delta^{\hat{\beta}}_{\hat{\tau})}\,, \quad (M_{\hat{\rho}\hat{\tau}})_{\hat{a}}{}^{\hat{b}} = (\Sigma_{\hat{a}}{}^{\hat{b}})_{\hat{\rho}\hat{\tau}}\,, \qquad (J_{kl})_{\underline{\hat{\alpha}}}{}^{\underline{\hat{\beta}}} = -\delta^{\hat{\beta}}_{\hat{\alpha}}\delta^i_{(k}\varepsilon_{l)j}\,, \tag{2.10c}$$

with the other components of $(M_{\hat{\rho}\hat{\tau}})_{\hat{C}}{}^{\hat{D}}$ and $(J_{kl})_{\hat{C}}{}^{\hat{D}}$ being equal to zero.

Similar to the well-known case of four-dimensional $\mathcal{N} = 1$ supergravity (see [22–24] for comprehensive reviews), the geometric superfields in (2.1) contain too many component fields to describe an irreducible supergravity multiplet. This can be cured by imposing covariant algebraic constraints on the geometry of superspace. In accordance with a theorem due to Dragon [25], it is sufficient to impose constraints on the torsion, since the curvature is completely determined in terms of the torsion in supergravity theories formulated in superspace.

As demonstrated in [20], in order to realize the minimal supergravity multiplet in the above framework, one has to impose the following constraints on various components of the torsion of dimensions 0, 1/2 and 1:

$$T^{i\,j\,\hat{c}}_{\hat{\alpha}\,\hat{\beta}} = -2\mathrm{i}\,\varepsilon^{ij}(\Gamma^{\hat{c}})_{\hat{\alpha}\hat{\beta}}\,, \quad F^{i\,j}_{\hat{\alpha}\,\hat{\beta}} = -2\mathrm{i}\,\varepsilon^{ij}\varepsilon_{\hat{\alpha}\hat{\beta}}\,, \qquad \text{(dimension 0)} \tag{2.11a}$$

$$T^{i\,j\,\hat{\gamma}}_{\hat{\alpha}\,\hat{\beta}k} = T^{i}_{\hat{\alpha}\hat{b}}{}^{\hat{c}} = F^{i}_{\hat{\alpha}\hat{b}} = 0\,, \qquad \text{(dimension 1/2)} \tag{2.11b}$$

$$T_{\hat{a}\hat{b}}{}^{\hat{c}} = T_{\hat{a}}{}^{l}_{\hat{\beta}}{}^{\hat{\beta}}_{(j}\,\varepsilon_{k)l} \;=\; 0\,. \qquad \text{(dimension 1)} \tag{2.11c}$$

Under these constraints, the Bianchi identities (2.9a), (2.9c) become non-trivial equations that have to be solved in order to determine the non-vanishing components of the torsion.

[6]The reader should keep in mind that we often use the condensed notation: $A_{\underline{\hat{\alpha}}} \equiv A^i_{\hat{\alpha}}$ and $A^{\underline{\hat{\alpha}}} \equiv A^{\hat{\alpha}}_i$.

2.1 The algebra of covariant derivatives

In this subsection, we summarize the results of the solution to the Bianchi identities based on the constraints (2.11a)–(2.11c), while the technical details will be given in the remainder of this section.

The algebra of covariant derivatives has the form [1]

$$
\begin{aligned}
\{\mathcal{D}^i_{\hat\alpha},\mathcal{D}^j_{\hat\beta}\} &= -2\mathrm{i}\,\varepsilon^{ij}\mathcal{D}_{\hat\alpha\hat\beta}-2\mathrm{i}\,\varepsilon^{ij}\varepsilon_{\hat\alpha\hat\beta}Z \\
&\quad +3\mathrm{i}\,\varepsilon_{\hat\alpha\hat\beta}\varepsilon^{ij}S^{kl}J_{kl}-2\mathrm{i}(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}\Big(F_{\hat a\hat b}+N_{\hat a\hat b}\Big)J^{ij} \\
&\quad -\mathrm{i}\,\varepsilon_{\hat\alpha\hat\beta}\varepsilon^{ij}F^{\hat c\hat d}M_{\hat c\hat d}+\frac{\mathrm{i}}{4}\varepsilon^{ij}\varepsilon^{\hat a\hat b\hat c\hat d\hat e}N_{\hat a\hat b}(\Gamma_{\hat c})_{\hat\alpha\hat\beta}M_{\hat d\hat e}+4\mathrm{i}\,S^{ij}M_{\hat\alpha\hat\beta}\,, \qquad (2.12\mathrm{a})\\
[\mathcal{D}_{\hat a},\mathcal{D}^j_{\hat\beta}] &= \frac12(\Gamma_{\hat a})_{\hat\beta}{}^{\hat\gamma}S^j{}_k\mathcal{D}^k_{\hat\gamma}-\frac12 F_{\hat a\hat b}(\Gamma^{\hat b})_{\hat\beta}{}^{\hat\gamma}\mathcal{D}^j_{\hat\gamma}-\frac18\varepsilon_{\hat a\hat b\hat c\hat d\hat e}N^{\hat d\hat e}(\Sigma^{\hat b\hat c})_{\hat\beta}{}^{\hat\gamma}\mathcal{D}^j_{\hat\gamma} \\
&\quad +\Big(-3\varepsilon^{jk}\Xi_{\hat a\hat\beta}{}^l+\frac54(\Gamma_{\hat a})_{\hat\beta}{}^{\hat\alpha}\varepsilon^{jk}\mathcal{F}_{\hat\alpha}{}^l-\frac14(\Gamma_{\hat a})_{\hat\beta}{}^{\hat\alpha}\varepsilon^{jk}\mathcal{N}_{\hat\alpha}{}^l\Big)J_{kl} \\
&\quad +\Big(\frac12(\Gamma_{\hat a})_{\hat\beta}{}^{\hat\rho}\mathcal{D}^j_{\hat\rho}F^{\hat c\hat d}-\frac12(\Gamma^{\hat c})_{\hat\beta}{}^{\hat\rho}\mathcal{D}^j_{\hat\rho}F^{\hat d}{}_{\hat a}+\frac12(\Gamma^{\hat d})_{\hat\beta}{}^{\hat\rho}\mathcal{D}^j_{\hat\rho}F^{\hat c}{}_{\hat a}\Big)M_{\hat c\hat d}\,, \qquad (2.12\mathrm{b})\\
[\mathcal{D}_{\hat a},\mathcal{D}_{\hat b}] &= \frac{\mathrm{i}}{2}\Big(\mathcal{D}^{\hat\gamma}_kF_{\hat a\hat b}\Big)\mathcal{D}^k_{\hat\gamma}-\frac{\mathrm{i}}{8}\Big(\mathcal{D}^{\hat\gamma(k}\mathcal{D}^{l)}_{\hat\gamma}F_{\hat a\hat b}\Big)J_{kl}+F_{\hat a\hat b}Z \\
&\quad +\Big(\frac14\varepsilon^{\hat c\hat d}{}_{\hat m\hat n[\hat a}\mathcal{D}_{\hat b]}N^{\hat m\hat n}+\frac12\delta^{\hat c}_{[\hat a}N_{\hat b]\hat m}N^{\hat d\hat m}-\frac14N_{\hat a}{}^{\hat c}N_{\hat b}{}^{\hat d}-\frac18\delta^{\hat c}_{\hat a}\delta^{\hat d}_{\hat b}N^{\hat m\hat n}N_{\hat m\hat n} \\
&\qquad +\frac{\mathrm{i}}{8}(\Sigma^{\hat c\hat d})^{\hat\gamma\hat\delta}\mathcal{D}^k_{\hat\gamma}\mathcal{D}_{\hat\delta k}F_{\hat a\hat b}-F_{\hat a}{}^{\hat c}F_{\hat b}{}^{\hat d}+\frac12\delta^{\hat c}_{\hat a}\delta^{\hat d}_{\hat b}S^{ij}S_{ij}\Big)M_{\hat c\hat d}\,. \qquad (2.12\mathrm{c})
\end{aligned}
$$

The components of the torsion in (2.12a)–(2.12c) obey further constraints implied by the Bianchi identities, some of which can be conveniently expressed in terms of the three irreducible components of $\mathcal{D}^k_{\hat\gamma}F_{\hat\alpha\hat\beta}$: a completely symmetric third-rank tensor $W_{\hat\alpha\hat\beta\hat\gamma}{}^k$, a gamma-traceless spin-vector $\Xi_{\hat a\,\hat\gamma}{}^k$ and a spinor $\mathcal{F}_{\hat\gamma}{}^k$. These components originate as follows:

$$
\begin{aligned}
\mathcal{D}^k_{\hat\gamma}F_{\hat\alpha\hat\beta} &= W_{\hat\alpha\hat\beta\hat\gamma}{}^k+\Xi_{\hat\gamma(\hat\alpha\hat\beta)}{}^k+\varepsilon_{\hat\gamma(\hat\alpha}\mathcal{F}_{\hat\beta)}{}^k\,,\\
\Xi_{\hat\gamma\hat\alpha\hat\beta}{}^k &= (\Gamma_{\hat a})_{\hat\gamma\hat\alpha}\Xi^{\hat a}{}_{\hat\beta}{}^k\,,\quad (\Gamma^{\hat a})_{\hat\alpha}{}^{\hat\beta}\Xi_{\hat a\hat\beta}{}^i=0\,,\quad W_{\hat\alpha\hat\beta\hat\gamma}{}^k=W_{(\hat\alpha\hat\beta\hat\gamma)}{}^k\,. \qquad (2.13)
\end{aligned}
$$

It is useful to have eq. (2.13) rewritten in the equivalent form ($W_{\hat a\hat b\hat\gamma}{}^k=(\Sigma^{\hat a\hat b})^{\hat\alpha\hat\beta}W_{\hat\alpha\hat\beta\hat\gamma}{}^k$)

$$
\mathcal{D}^k_{\hat\gamma}F_{\hat a\hat b} = W_{\hat a\hat b\hat\gamma}{}^k+2(\Gamma_{[\hat a})_{\hat\gamma}{}^{\hat\delta}\Xi_{\hat b]\hat\delta}{}^k+(\Sigma_{\hat a\hat b})_{\hat\gamma}{}^{\hat\delta}\mathcal{F}_{\hat\delta}{}^k\,,\quad (\Gamma^{\hat a})_{\hat\alpha}{}^{\hat\gamma}W_{\hat a\hat b\hat\gamma}{}^i=0\,. \qquad (2.14)
$$

The dimension 3/2 Bianchi identities are as folllows:

$$
\begin{aligned}
\mathcal{D}^k_{\hat\gamma}N_{\hat\alpha\hat\beta} &= -W_{\hat\alpha\hat\beta\hat\gamma}{}^k+2\,\Xi_{\hat\gamma(\hat\alpha\hat\beta)}{}^k+\varepsilon_{\hat\gamma(\hat\alpha}\mathcal{N}_{\hat\beta)}{}^k\,, \qquad (2.15\mathrm{a})\\
\mathcal{D}^k_{\hat\beta}S^{jl} &= \frac{1}{10}(\Sigma_{\hat a\hat b})_{\hat\beta}{}^{\hat\delta}\varepsilon^{k(j}\mathcal{D}^{l)}_{\hat\delta}\Big(3F^{\hat a\hat b}+N^{\hat a\hat b}\Big) = -\frac12\varepsilon^{k(j}\Big(3\mathcal{F}_{\hat\beta}{}^{l)}+\mathcal{N}_{\hat\beta}{}^{l)}\Big)\,. \qquad (2.15\mathrm{b})
\end{aligned}
$$

Equation (2.15a) can be equivalently expressed in the form

$$
\mathcal{D}^k_{\hat\gamma}N_{\hat a\hat b} = -W_{\hat a\hat b\hat\gamma}{}^k+4(\Gamma_{[\hat a})_{\hat\gamma}{}^{\hat\delta}\Xi_{\hat b]\hat\delta}{}^k+(\Sigma_{\hat a\hat b})_{\hat\gamma}{}^{\hat\delta}\mathcal{N}_{\hat\delta}{}^k\,. \qquad (2.16)
$$

The dimension 2 Bianchi identities are:

$$
\begin{aligned}
\mathcal{D}^{(k}_{[\hat\beta}\mathcal{N}_{\hat\delta]}{}^{l)} &= -\mathcal{D}^{(k}_{[\hat\beta}\mathcal{F}_{\hat\delta]}{}^{l)} - \frac{3}{4}\mathcal{D}^{\hat\gamma(k}\Xi_{\hat\beta\hat\delta\hat\gamma}{}^{l)}\,, && (2.17a)\\
(\Gamma_{\hat a})^{\hat\alpha\hat\beta}\mathcal{D}^k_{\hat\alpha}\mathcal{N}_{\hat\beta k} &= (\Gamma_{\hat a})^{\hat\alpha\hat\beta}\mathcal{D}^k_{\hat\alpha}\mathcal{F}_{\hat\beta k} + 4\mathcal{D}^{\hat\alpha}_k\Xi_{\hat a\hat\alpha}{}^k - \frac{4\mathrm{i}}{3}\varepsilon_{\hat a\hat m\hat n\hat p\hat q}\Big(N^{\hat m\hat n}N^{\hat p\hat q} + F^{\hat m\hat n}F^{\hat p\hat q}\Big)\,, && (2.17b)\\
(\Sigma^{\hat a\hat b})^{\hat\alpha\hat\beta}\mathcal{D}^k_{\hat\alpha}\mathcal{N}_{\hat\beta k} &= -5(\Sigma^{\hat a\hat b})^{\hat\alpha\hat\beta}\mathcal{D}^k_{\hat\alpha}\mathcal{F}_{\hat\beta k} + 6(\Gamma^{[\hat a})^{\hat\alpha\hat\beta}\mathcal{D}^k_{\hat\alpha}\Xi^{\hat b]}{}_{\hat\beta k} + 16\mathrm{i}F_{\hat m}{}^{[\hat a}N^{\hat b]\hat m}\,, && (2.17c)\\
\mathcal{D}^{(k}_{[\hat\alpha}W_{\hat\beta]\hat\gamma\hat\delta}{}^{l)} &= \frac{1}{2}\varepsilon_{\hat\alpha\hat\beta}\mathcal{D}^{(k}_{(\hat\gamma}\mathcal{F}_{\hat\delta)}{}^{l)} - \frac{1}{2}\mathcal{D}^{(k}_{[\hat\alpha}\varepsilon_{\hat\beta](\hat\gamma}\mathcal{F}_{\hat\delta)}{}^{l)} - \frac{1}{2}\mathcal{D}^{(k}_{(\hat\gamma}\varepsilon_{\hat\delta)[\hat\alpha}\mathcal{F}_{\hat\beta]}{}^{l)}\\
&\quad + \frac{3}{4}\varepsilon_{\hat\alpha\hat\beta}\mathcal{D}^{\hat\rho(k}\Xi_{\hat\rho(\hat\gamma\hat\delta)}{}^{l)} - \frac{3}{16}\varepsilon_{\hat\gamma[\hat\alpha}\mathcal{D}^{\hat\rho(k}\Xi_{\hat\beta]\hat\delta\hat\rho}{}^{l)} - \frac{3}{16}\varepsilon_{\hat\delta[\hat\alpha}\mathcal{D}^{\hat\rho(k}\Xi_{\hat\beta]\hat\gamma\hat\rho}{}^{l)}\\
&\quad - \mathcal{D}^{(k}_{[\hat\alpha}\Xi_{\hat\beta](\hat\gamma\hat\delta)}{}^{l)} + 2\mathrm{i}\Big(\varepsilon_{\hat\alpha\hat\beta}N_{\hat\gamma\hat\delta} - \varepsilon_{\hat\gamma[\hat\alpha}N_{\hat\beta]\hat\delta} - \varepsilon_{\hat\delta[\hat\alpha}N_{\hat\beta]\hat\gamma}\Big)S^{kl}\,, && (2.17d)\\
0 &= \mathcal{D}_{\hat a}F_{\hat b\hat c} + \mathcal{D}_{\hat b}F_{\hat c\hat a} + \mathcal{D}_{\hat c}F_{\hat a\hat b}\,. && (2.17e)
\end{aligned}
$$

Note that eq. (2.17e) can equivalently be rewritten as

$$0 = \mathcal{D}^{\hat\alpha}_k W^{\hat a\hat b}{}_{\hat\alpha}{}^k + 2\varepsilon^{\hat a\hat b\hat c\hat d\hat e}(\Sigma_{\hat c\hat d})^{\hat\alpha\hat\beta}\mathcal{D}^k_{\hat\alpha}\Xi_{\hat e\hat\beta k} - 3(\Sigma^{\hat a\hat b})^{\hat\alpha\hat\beta}\mathcal{D}^k_{\hat\alpha}\mathcal{F}_{\hat\beta k} + 16\mathrm{i}F_{\hat c}{}^{[\hat a}N^{\hat b]\hat c}\,. \qquad (2.18)$$

2.2 Solving the Bianchi identities: dimension 1

Now, we turn to solving the Bianchi identities (2.9a)–(2.9c) based on the constraints (2.11a)–(2.11c). It is standard and useful to organize the analysis in accordance with the increasing dimension of the identities involved (from dimension 1/2 to 3).

The important simplification is that it is sufficient to analyze only the Bianchi identities (2.9a) and (2.9c), due to Dragon's second theorem [25]. The latter states that all the equations (2.9b) are identically satisfied, provided (2.9a) and (2.9c) hold.

For dimension 1/2, the relations (2.9a) with ($\hat A = \underline{\hat\alpha},\ \hat B = \underline{\hat\beta},\ \hat C = \underline{\hat\gamma},\ \hat D = \hat d$) and (2.9c) with ($\hat A = \underline{\hat\alpha},\ \hat B = \underline{\hat\beta},\ \hat C = \underline{\hat\gamma}$) are identically satisfied.

For dimension 1, there occur several Bianchi identities that originate from eqs. (2.9a) and (2.9c). Setting ($\hat A = \hat a,\ \hat B = \underline{\hat\beta},\ \hat C = \underline{\hat\gamma},\ \hat D = \hat d$) in (2.9a) gives

$$0 = R^{j\,k}_{\hat\beta\hat\gamma\hat a}{}^{\hat d} + 2\mathrm{i}T_{\hat a}{}^{j}_{\hat\beta}{}^{\hat\rho k}(\Gamma^{\hat d})_{\hat\rho\hat\gamma} + 2\mathrm{i}T_{\hat a}{}^{k}_{\hat\gamma}{}^{\hat\rho j}(\Gamma^{\hat d})_{\hat\rho\hat\beta}\,, \qquad (2.19)$$

while the choice ($\hat A = \underline{\hat\alpha},\ \hat B = \underline{\hat\beta},\ \hat C = \underline{\hat\gamma},\ \hat D = \underline{\hat\delta}$) leads to

$$
\begin{aligned}
0 = \ & R^{i\,j}_{\hat\alpha\hat\beta\hat\gamma}{}^{\hat\delta}\delta^k_l + R^{j\,k}_{\hat\beta\hat\gamma\hat\alpha}{}^{\hat\delta}\delta^i_l + R^{k\,i}_{\hat\gamma\hat\alpha\hat\beta}{}^{\hat\delta}\delta^j_l + R^{i\,j\,k}_{\hat\alpha\hat\beta}{}_l\delta^{\hat\delta}_{\hat\gamma} + R^{j\,k\,i}_{\hat\beta\hat\gamma}{}_l\delta^{\hat\delta}_{\hat\alpha} + R^{k\,i\,j}_{\hat\gamma\hat\alpha}{}_l\delta^{\hat\delta}_{\hat\beta}\\
& - 2\mathrm{i}\varepsilon^{ij}(\Gamma^{\hat e})_{\hat\alpha\hat\beta}T_{\hat e}{}^{k\,\hat\delta}_{\hat\gamma\,l} - 2\mathrm{i}\varepsilon^{jk}(\Gamma^{\hat e})_{\hat\beta\hat\gamma}T_{\hat e}{}^{i\,\hat\delta}_{\hat\alpha\,l} - 2\mathrm{i}\varepsilon^{ki}(\Gamma^{\hat e})_{\hat\gamma\hat\alpha}T_{\hat e}{}^{j\,\hat\delta}_{\hat\beta\,l}\,. \qquad (2.20)
\end{aligned}
$$

Choosing ($\hat A = \hat a,\ \hat B = \underline{\hat\beta},\ \hat C = \underline{\hat\gamma}$) in (2.9c) gives

$$0 = T_{\hat a}{}^{j\,k}_{\hat\beta\hat\gamma} + \varepsilon^{jk}(\Gamma^{\hat d})_{\hat\beta\hat\gamma}F_{\hat d\hat a} + T_{\hat a}{}^{k\,j}_{\hat\gamma\hat\beta}\,. \qquad (2.21)$$

Eq. (2.21) implies that the dimension 1 torsion can be represented in the form:

$$T_{\hat a}{}^{j\,k}_{\hat\beta\hat\gamma} = \frac{1}{2}\varepsilon^{jk}(\Gamma^{\hat b})_{\hat\beta\hat\gamma}F_{\hat a\hat b} - \frac{1}{4}\varepsilon^{jk}(\Sigma^{\hat b\hat c})_{\hat\beta\hat\gamma}T_{1\hat a\hat b\hat c} + \frac{1}{2}(\Gamma^{\hat b})_{\hat\beta\hat\gamma}T_{1\hat a\hat b}{}^{jk} - \frac{1}{4}\varepsilon_{\hat\beta\hat\gamma}T_{1\hat a}{}^{jk}\,, \qquad (2.22)$$

where

$$T_{1\hat{a}}{}^{jk} = T_{1\hat{a}}{}^{kj}\,, \qquad T_{1\hat{a}\hat{b}}{}^{jk} = T_{1\hat{a}\hat{b}}{}^{kj}\,, \qquad T_{1\hat{a}\hat{b}\hat{c}} = -T_{1\hat{a}\hat{c}\hat{b}}\,. \tag{2.23}$$

Equation (2.19) expresses the dimension 1 Lorentz curvature in terms of the torsion

$$R_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j}{}^{\hat{c}\hat{d}} = -2\mathrm{i}T^{\hat{c}i}{}_{\hat{\alpha}}{}^{\hat{\rho}j}(\Gamma^{\hat{d}})_{\hat{\rho}\hat{\beta}} - 2\mathrm{i}T^{\hat{c}j}{}_{\hat{\beta}}{}^{\hat{\rho}i}(\Gamma^{\hat{d}})_{\hat{\rho}\hat{\alpha}}\,. \tag{2.24}$$

Since $R_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j}{}^{\hat{c}\hat{d}} = -R_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j}{}^{\hat{d}\hat{c}}$, the following equation occurs

$$0 = (\Gamma^{(\hat{a}})_{\hat{\rho}\hat{\gamma}}T^{\hat{d})j}{}_{\hat{\beta}}{}^{\hat{\rho}k} + (\Gamma^{(\hat{a}})_{\hat{\rho}\hat{\beta}}T^{\hat{d})k}{}_{\hat{\gamma}}{}^{\hat{\rho}j} = \frac{1}{2}(\Gamma^{\hat{c}})_{\hat{\beta}\hat{\gamma}}\varepsilon^{jk}T_1{}^{(\hat{a}\hat{d})}{}_{\hat{c}} - 2(\Sigma^{\hat{b}(\hat{a}})_{\hat{\beta}\hat{\gamma}}T_1{}^{\hat{d})}{}_{\hat{b}}{}^{jk}\,. \tag{2.25}$$

This holds if and only if $T_{1\hat{a}\hat{b}}{}^{kl}$ and $T_{1\hat{a}\hat{b}\hat{c}}$ have the form:

$$\begin{aligned} T_{1\hat{a}\hat{b}}{}^{ij} &= \frac{1}{5}\eta_{\hat{a}\hat{b}}\eta^{\hat{m}\hat{n}}T_{1\hat{m}\hat{n}}{}^{ij} \equiv \eta_{\hat{a}\hat{b}}S^{ij}\,, &\qquad S^{ij} &= S^{ji}\,, \\ T_{1\hat{a}\hat{b}\hat{c}} &= -T_{1\hat{b}\hat{a}\hat{c}} \equiv N_{\hat{a}\hat{b}\hat{c}}\,, & N_{\hat{a}\hat{b}\hat{c}} &= N_{[\hat{a}\hat{b}\hat{c}]}\,, \end{aligned} \tag{2.26}$$

for some symmetric tensor S^{ij} obeying the reality condition $\overline{S^{ij}} = S_{ij}$, and a completely antisymmetric real tensor $N_{\hat{a}\hat{b}\hat{c}}$. As a result, the Lorentz curvature (2.24) takes the form:

$$R_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j}{}^{\hat{c}\hat{d}} = -\mathrm{i}\varepsilon_{\hat{\alpha}\hat{\beta}}\varepsilon^{ij}F^{\hat{c}\hat{d}} - \frac{\mathrm{i}}{2}\varepsilon^{ij}N^{\hat{c}\hat{d}\hat{e}}(\Gamma_{\hat{e}})_{\hat{\alpha}\hat{\beta}} + 4\mathrm{i}S^{ij}(\Sigma^{\hat{c}\hat{d}})_{\hat{\alpha}\hat{\beta}}\,. \tag{2.27}$$

Let us now turn to eq. (2.20). Taking the trace over the indices $\hat{\gamma}$ and $\hat{\delta}$, one derive the following equation for the SU(2)-curvature:

$$4R_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j}{}^{kl} + R_{\hat{\alpha}}^{k}{}_{\hat{\beta}}^{j}{}^{il} + R_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{k}{}^{jl} = \Delta_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j}{}^{kl}\,, \tag{2.28}$$

with

$$\Delta_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j}{}^{kl} = 15\mathrm{i}\varepsilon_{\hat{\alpha}\hat{\beta}}\varepsilon^{ij}S^{kl} + \frac{5\mathrm{i}}{2}(\Gamma^{\hat{c}})_{\hat{\alpha}\hat{\beta}}\varepsilon^{ij}T_{1\hat{c}}{}^{kl} + \mathrm{i}(\Sigma^{\hat{d}\hat{e}})_{\hat{\alpha}\hat{\beta}}\varepsilon^{k(i}\varepsilon^{j)l}\Big(6F_{\hat{d}\hat{e}} + \varepsilon_{\hat{d}\hat{e}\hat{a}\hat{b}\hat{c}}N^{\hat{a}\hat{b}\hat{c}}\Big)\,. \tag{2.29}$$

Here we have used the explicit expressions for the dimension 1 torsion and for the Lorentz curvature in terms of S^{kl}, $T_{1\hat{a}}{}^{jk}$ and $N_{\hat{a}\hat{b}\hat{c}}$.

Equation (2.28) allows us to express $R_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j}{}^{kl}$ in terms of $\Delta_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j}{}^{kl}$, and the result is

$$R_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j}{}^{kl} = \frac{1}{90}\Big(26\Delta_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j}{}^{kl} - \Delta_{\hat{\alpha}}^{j}{}_{\hat{\beta}}^{i}{}^{kl} + 2\Delta_{\hat{\alpha}}^{k}{}_{\hat{\beta}}^{i}{}^{jl} - 7\Delta_{\hat{\alpha}}^{k}{}_{\hat{\beta}}^{j}{}^{il} - 7\Delta_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{k}{}^{jl} + 2\Delta_{\hat{\alpha}}^{j}{}_{\hat{\beta}}^{k}{}^{il}\Big)\,. \tag{2.30}$$

It is useful to introduce the Hodge-dual of $N_{\hat{a}\hat{b}\hat{c}}$, $N_{\hat{a}\hat{b}} \equiv \frac{1}{6}\varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}N^{\hat{c}\hat{d}\hat{e}}$. Then, the SU(2)-curvature can be rewritten in the form:

$$R_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j}{}^{kl} = 3\mathrm{i}\varepsilon_{\hat{\alpha}\hat{\beta}}\varepsilon^{ij}S^{kl} + \frac{\mathrm{i}}{2}(\Gamma^{\hat{c}})_{\hat{\alpha}\hat{\beta}}\varepsilon^{ij}T_{1\hat{c}}{}^{kl} + 2\mathrm{i}(\Sigma^{\hat{a}\hat{b}})_{\hat{\alpha}\hat{\beta}}\Big(F_{\hat{a}\hat{b}} + N_{\hat{a}\hat{b}}\Big)\varepsilon^{k(i}\varepsilon^{j)l}\,. \tag{2.31}$$

Using the results obtained and the fact that the constraint (2.11c) is equivalent to

$$T_{1\hat{a}}{}^{jk} = 0\,, \tag{2.32}$$

eq. (2.20) is now solved, and the dimension 1 torsion becomes

$$T_{\hat{a}}{}_{\hat{\beta}}^{j}{}_{\hat{\gamma}}^{k} = \frac{1}{2}(\Gamma_{\hat{a}})_{\hat{\beta}\hat{\gamma}}S^{jk} + \frac{1}{2}\varepsilon^{jk}(\Gamma^{\hat{b}})_{\hat{\beta}\hat{\gamma}}F_{\hat{a}\hat{b}} - \frac{1}{4}\varepsilon^{jk}(\Sigma^{\hat{b}\hat{c}})_{\hat{\beta}\hat{\gamma}}N_{\hat{a}\hat{b}\hat{c}}\,. \tag{2.33}$$

The final form for the SU(2)-curvature is

$$R_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j}{}^{kl} = 3\mathrm{i}\varepsilon_{\hat{\alpha}\hat{\beta}}\varepsilon^{ij}S^{kl} + 2\mathrm{i}(\Sigma^{\hat{a}\hat{b}})_{\hat{\alpha}\hat{\beta}}\Big(F_{\hat{a}\hat{b}} + N_{\hat{a}\hat{b}}\Big)\varepsilon^{k(i}\varepsilon^{j)l}\,. \tag{2.34}$$

2.3 Solving the Bianchi identities: dimension 3/2

For dimension 3/2, the relevant Bianchi identities come from both equations (2.9a) and (2.9c). Setting $(\hat{A}=\hat{a},\ \hat{B}=\underline{\hat{\beta}},\ \hat{C}=\underline{\hat{\gamma}},\ \hat{D}=\underline{\hat{\delta}})$ in eq. (2.9a) gives

$$\begin{aligned} 0 = \ & R_{\hat{a}}{}^{j}_{\hat{\beta}}{}_{\hat{\gamma}}{}^{\hat{\delta}}\delta^k_l + R_{\hat{a}}{}^{k}_{\hat{\gamma}}{}_{\hat{\beta}}{}^{\hat{\delta}}\delta^j_l + R_{\hat{a}}{}^{j}_{\hat{\beta}}{}^{k}{}_{l}\delta^{\hat{\delta}}_{\hat{\gamma}} + R_{\hat{a}}{}^{k}_{\hat{\gamma}}{}^{j}{}_{l}\delta^{\hat{\delta}}_{\hat{\beta}} \\ & + \mathcal{D}^j_{\hat{\beta}} T_{\hat{a}}{}^{k}_{\hat{\gamma}}{}^{\hat{\delta}}_{l} + 2\mathrm{i}\varepsilon^{jk}(\Gamma^{\hat{e}})_{\hat{\beta}\hat{\gamma}} T_{\hat{a}\hat{e}}{}^{\hat{\delta}}_{l} + \mathcal{D}^k_{\hat{\gamma}} T_{\hat{a}}{}^{j}_{\hat{\beta}}{}^{\hat{\delta}}_{l} , \end{aligned} \tag{2.35}$$

while the choice $(\hat{A}=\hat{a},\ \hat{B}=\hat{b},\ \hat{C}=\underline{\hat{\gamma}},\ \hat{D}=\hat{d})$ in (2.9a) results in

$$0 = R_{\hat{b}}{}^{k}_{\hat{\gamma}}{}_{\hat{a}}{}^{\hat{d}} - R_{\hat{a}}{}^{k}_{\hat{\gamma}}{}_{\hat{b}}{}^{\hat{d}} + 2\mathrm{i}T_{\hat{a}\hat{b}}{}^{\hat{\rho}k}(\Gamma^{\hat{d}})_{\hat{\rho}\hat{\gamma}} . \tag{2.36}$$

Choosing $(\hat{A}=\hat{a},\ \hat{B}=\hat{b},\ \hat{C}=\underline{\hat{\gamma}})$ in eq. (2.9c) gives

$$0 = 2\mathrm{i}T_{\hat{a}\hat{b}}{}^{k}_{\hat{\gamma}} + \mathcal{D}^k_{\hat{\gamma}} F_{\hat{a}\hat{b}} . \tag{2.37}$$

For the analysis of the above identities, it is advantageous to make use of the decomposition of a spin-tensor $A_{\hat{a}\hat{b}\hat{\gamma}} = -A_{\hat{b}\hat{a}\hat{\gamma}}$ into its irreducible components:

$$A_{\hat{a}\hat{b}\hat{\gamma}} = \mathcal{A}_{\hat{a}\hat{b}\hat{\gamma}} + 2(\Gamma_{[\hat{a}})_{\hat{\gamma}}{}^{\hat{\delta}}\mathcal{A}_{\hat{b}]\hat{\delta}} + (\Sigma_{\hat{a}\hat{b}})_{\hat{\gamma}}{}^{\hat{\delta}}\mathcal{A}_{\hat{\delta}} , \quad (\Gamma^{\hat{a}})_{\hat{\alpha}}{}^{\hat{\gamma}}\mathcal{A}_{\hat{a}\hat{\gamma}} = (\Gamma^{\hat{a}})_{\hat{\alpha}}{}^{\hat{\gamma}}\mathcal{A}_{\hat{a}\hat{b}\hat{\gamma}} = 0 . \tag{2.38a}$$

Switching to the spinor notations, we have have to deal with

$$A_{\hat{\alpha}\hat{\beta}\hat{\gamma}} := \frac{1}{2}(\Sigma^{\hat{a}\hat{b}})_{\hat{\alpha}\hat{\beta}} A_{\hat{a}\hat{b}\hat{\gamma}} = A_{(\hat{\alpha}\hat{\beta})\hat{\gamma}} , \tag{2.39}$$

and the corresponding decomposition is

$$\begin{aligned} & A_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = \mathcal{A}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} + \widetilde{\mathcal{A}}_{\hat{\gamma}(\hat{\alpha}\hat{\beta})} + \varepsilon_{\hat{\gamma}(\hat{\alpha}}\mathcal{A}_{\hat{\beta})} , \qquad \mathcal{A}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = \mathcal{A}_{(\hat{\alpha}\hat{\beta}\hat{\gamma})} , \\ & \widetilde{\mathcal{A}}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} := (\Gamma_{\hat{a}})_{\hat{\alpha}\hat{\beta}}\mathcal{A}^{\hat{a}}{}_{\hat{\gamma}} = -\widetilde{\mathcal{A}}_{\hat{\beta}\hat{\alpha}\hat{\gamma}} , \qquad \varepsilon^{\hat{\alpha}\hat{\beta}}\widetilde{\mathcal{A}}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = \varepsilon^{\hat{\alpha}\hat{\gamma}}\widetilde{\mathcal{A}}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = 0 , \quad \widetilde{\mathcal{A}}_{[\hat{\alpha}\hat{\beta}\hat{\gamma}]} = 0 . \end{aligned} \tag{2.40}$$

From equation (2.37) we immediately read off the dimension 3/2 torsion

$$T_{\hat{a}\hat{b}k}{}^{\hat{\gamma}} = \frac{\mathrm{i}}{2}\mathcal{D}^{\hat{\gamma}}_{k} F_{\hat{a}\hat{b}} . \tag{2.41}$$

Applying the decomposition (2.40) to the right-hand side of (2.41) gives

$$\mathcal{D}^k_{\hat{\gamma}} F_{\hat{\alpha}\hat{\beta}} = W_{\hat{\alpha}\hat{\beta}\hat{\gamma}}{}^{k} + \Xi_{\hat{\gamma}(\hat{\alpha}\hat{\beta})}{}^{k} + \varepsilon_{\hat{\gamma}(\hat{\alpha}}\mathcal{F}_{\hat{\beta})}{}^{k} . \tag{2.42}$$

Next, equation (2.36) is solved by

$$R_{\hat{a}}{}^{j}_{\hat{\beta}}{}^{\hat{c}\hat{d}} = \frac{1}{2}(\Gamma_{\hat{a}})_{\hat{\beta}}{}^{\hat{\delta}}\mathcal{D}^j_{\hat{\delta}}F^{\hat{c}\hat{d}} - \frac{1}{2}(\Gamma^{\hat{c}})_{\hat{\beta}}{}^{\hat{\delta}}\mathcal{D}^j_{\hat{\delta}}F^{\hat{d}}{}_{\hat{a}} + \frac{1}{2}(\Gamma^{\hat{d}})_{\hat{\beta}}{}^{\hat{\delta}}\mathcal{D}^j_{\hat{\delta}}F^{\hat{c}}{}_{\hat{a}} . \tag{2.43a}$$

Equation (2.35) allows us to compute the SU(2)-curvature $R_{\hat{a}}{}^{j}_{\hat{\beta}}{}^{kl}$. Taking the trace over $\hat{\gamma}$ and $\hat{\delta}$ in eq. (2.35) gives

$$4R_{\hat{a}}{}^{j}_{\hat{\beta}}{}^{kl} + R_{\hat{a}}{}^{k}_{\hat{\beta}}{}^{jl} = \Delta_{\hat{a}}{}^{j}_{\hat{\beta}}{}^{kl} , \tag{2.44}$$

where

$$\Delta_{\hat{a}\hat{\beta}}{}^{j\,kl} = -\frac{7}{4}(\Gamma^{\hat{b}})_{\hat{\beta}}{}^{\hat{\gamma}}\mathcal{D}^k_{\hat{\gamma}}F_{\hat{a}\hat{b}}\varepsilon^{jl} - \varepsilon^{jk}(\Gamma^{\hat{b}})_{\hat{\beta}}{}^{\hat{\gamma}}\mathcal{D}^l_{\hat{\gamma}}F_{\hat{a}\hat{b}} - \frac{1}{8}\varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}(\Sigma^{\hat{d}\hat{e}})_{\hat{\beta}}{}^{\hat{\gamma}}\mathcal{D}^k_{\hat{\gamma}}F^{\hat{b}\hat{c}}\varepsilon^{jl}$$
$$+\frac{1}{4}\varepsilon^{jl}(\Sigma^{\hat{b}\hat{c}})_{\hat{\beta}}{}^{\hat{\gamma}}\mathcal{D}^k_{\hat{\gamma}}N_{\hat{a}\hat{b}\hat{c}} - \frac{1}{2}(\Gamma_{\hat{a}})_{\hat{\beta}}{}^{\hat{\gamma}}\mathcal{D}^k_{\hat{\gamma}}S^{jl}\ . \tag{2.45}$$

Equation (2.44) is solved by

$$R_{\hat{a}\hat{\beta}}{}^{j\,kl} = \frac{4}{15}\Delta_{\hat{a}\hat{\beta}}{}^{j\,kl} - \frac{1}{15}\Delta_{\hat{a}\hat{\beta}}{}^{k\,jl}\ . \tag{2.46}$$

Since $R_{\hat{a}\hat{\beta}}{}^{j\,kl}$ is symmetric in k and l, eq. (2.46) can be seen to be consistent under the conditions:

$$\mathcal{D}_{\hat{\alpha}k}S^{kj} = \frac{3}{20}(\Sigma^{\hat{a}\hat{b}})_{\hat{\alpha}}{}^{\hat{\gamma}}\mathcal{D}^j_{\hat{\gamma}}\Big(3F_{\hat{a}\hat{b}} + N_{\hat{a}\hat{b}}\Big)\,, \tag{2.47a}$$
$$\mathcal{D}^k_{\hat{\gamma}}N_{\hat{\alpha}\hat{\beta}} = \mathcal{N}_{\hat{\alpha}\hat{\beta}\hat{\gamma}}{}^k + 2\,\Xi_{\hat{\gamma}(\hat{\alpha}\hat{\beta})}{}^k + \varepsilon_{\hat{\gamma}(\hat{\alpha}}\mathcal{N}_{\hat{\beta})}{}^k\ . \tag{2.47b}$$

Here $\Xi_{\hat{\alpha}\hat{\beta}}{}^k$ is the spin-vector which occurs in (2.42). At this point, the SU(2)-curvature has been completely determined.

$$R_{\hat{a}\hat{\beta}}{}^{j\,kl} = -\frac{4}{5}(\Gamma^{\hat{b}})_{\hat{\beta}}{}^{\hat{\gamma}}\varepsilon^{j(k}\mathcal{D}^{l)}_{\hat{\gamma}}F_{\hat{a}\hat{b}} - \frac{1}{30}\varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}(\Sigma^{\hat{d}\hat{e}})_{\hat{\beta}}{}^{\hat{\gamma}}\varepsilon^{j(k}\mathcal{D}^{l)}_{\hat{\gamma}}\Big(F^{\hat{b}\hat{c}} + N^{\hat{b}\hat{c}}\Big)$$
$$-\frac{2}{15}(\Gamma_{\hat{a}})_{\hat{\beta}}{}^{\hat{\gamma}}\mathcal{D}^{(k}_{\hat{\gamma}}S^{l)j} + \frac{1}{30}(\Gamma_{\hat{a}})_{\hat{\beta}}{}^{\hat{\gamma}}\mathcal{D}^j_{\hat{\gamma}}S^{kl}\ . \tag{2.48}$$

Using the previous results, one can prove that equation (2.35) implies the last two constraints:

$$\mathcal{N}_{\hat{\alpha}\hat{\beta}\hat{\gamma}}{}^k = -W_{\hat{\alpha}\hat{\beta}\hat{\gamma}}{}^k\,, \tag{2.49a}$$
$$\mathcal{D}^k_{\hat{\beta}}S^{jl} = \frac{1}{10}(\Sigma_{\hat{a}\hat{b}})_{\hat{\beta}}{}^{\hat{\delta}}\varepsilon^{k(j}\mathcal{D}^{l)}_{\hat{\delta}}\Big(3F^{\hat{a}\hat{b}} + N^{\hat{a}\hat{b}}\Big) = -\frac{1}{2}\varepsilon^{k(j}\Big(3\mathcal{F}_{\hat{\beta}}{}^{l)} + \mathcal{N}_{\hat{\beta}}{}^{l)}\Big)\ . \tag{2.49b}$$

It is important to note that (2.49b) implies equation (2.47a).

Expression (2.48) can actually be further simplified, using eqs. (2.14), (2.13), (2.16), (2.15a) and (2.15b). The final expression for the SU(2)-curvature is

$$R_{\hat{a}\hat{\beta}}{}^{j\,kl} = -3\varepsilon^{j(k}\Xi_{\hat{a}\hat{\beta}}{}^{l)} + \frac{5}{4}(\Gamma_{\hat{a}})_{\hat{\beta}}{}^{\hat{\alpha}}\varepsilon^{j(k}\mathcal{F}_{\hat{\alpha}}{}^{l)} - \frac{1}{4}(\Gamma_{\hat{a}})_{\hat{\beta}}{}^{\hat{\alpha}}\varepsilon^{j(k}\mathcal{N}_{\hat{\alpha}}{}^{l)}\ . \tag{2.50}$$

2.4 Solving the Bianchi identities: dimension 2

For dimension 2, the relevant Bianchi identities are generated from eq. (2.9a) with ($\hat{A} = \hat{a},\ \hat{B} = \hat{b},\ \hat{C} = \underline{\hat{\gamma}},\ \hat{D} = \underline{\hat{\delta}}$)

$$0 = R_{\hat{a}\hat{b}}{}^k{}_l\delta^{\hat{\delta}}_{\hat{\gamma}} + R_{\hat{a}\hat{b}\hat{\gamma}}{}^{\hat{\delta}}\delta^k_l - \mathcal{D}_{\hat{a}}T_{\hat{b}\hat{\gamma}l}^{k\hat{\delta}} + \mathcal{D}_{\hat{b}}T_{\hat{a}\hat{\gamma}l}^{k\hat{\delta}} - T_{\hat{b}\hat{\gamma}q}^{k\hat{\rho}}T_{\hat{a}\hat{\rho}l}^{q\hat{\delta}} - \mathcal{D}^k_{\hat{\gamma}}T_{\hat{a}\hat{b}l}^{\hat{\delta}} + T_{\hat{a}\hat{\gamma}q}^{k\hat{\rho}}T_{\hat{b}\hat{\rho}l}^{q\hat{\delta}}\,, \tag{2.51}$$

from (2.9a) with ($\hat{A} = \hat{a},\ \hat{B} = \hat{b},\ \hat{C} = \hat{c},\ \hat{D} = \hat{d}$)

$$0 = R_{\hat{a}\hat{b}\hat{c}}{}^{\hat{d}} + R_{\hat{b}\hat{c}\hat{a}}{}^{\hat{d}} + R_{\hat{c}\hat{a}\hat{b}}{}^{\hat{d}}\,, \tag{2.52}$$

and also from (2.9c) with ($\hat{A}=\hat{a},\ \hat{B}=\hat{b},\ \hat{C}=\hat{c}$)

$$0 = \mathcal{D}_{\hat{a}}F_{\hat{b}\hat{c}} + \mathcal{D}_{\hat{b}}F_{\hat{c}\hat{a}} + \mathcal{D}_{\hat{c}}F_{\hat{a}\hat{b}}\ . \tag{2.53}$$

Let us first analyze eq. (2.51). This can be used to extract the curvatures. We start by rewriting (2.51) in the form:

$$R_{\hat{a}\hat{b}}{}^{kl}\varepsilon_{\hat{\gamma}\hat{\delta}} + R_{\hat{a}\hat{b}\hat{\gamma}\hat{\delta}}\varepsilon^{kl} = \Delta_{\hat{a}\hat{b}\hat{\gamma}\hat{\delta}}^{\ kl}\,, \tag{2.54}$$

with

$$\begin{aligned}\Delta_{\hat{a}\hat{b}\hat{\gamma}\hat{\delta}}^{\ kl} =& -\frac{\mathrm{i}}{2}\mathcal{D}^k_{\hat{\gamma}}\mathcal{D}^l_{\hat{\delta}}F_{\hat{a}\hat{b}} - \varepsilon^{kl}(\Gamma^{\hat{c}})_{\hat{\gamma}\hat{\delta}}\mathcal{D}_{[\hat{a}}F_{\hat{b}]\hat{c}} + \frac{1}{4}\varepsilon^{kl}\varepsilon_{\hat{m}\hat{n}\hat{d}\hat{e}[\hat{a}}(\Sigma^{\hat{m}\hat{n}})_{\hat{\gamma}\hat{\delta}}\mathcal{D}_{\hat{b}]}N^{\hat{d}\hat{e}} + (\Gamma_{[\hat{a}})_{\hat{\gamma}\hat{\delta}}\mathcal{D}_{\hat{b}]}S^{kl}\\ &+\varepsilon^{kl}(\Sigma^{\hat{c}\hat{d}})_{\hat{\gamma}\hat{\delta}}F_{\hat{a}\hat{c}}F_{\hat{b}\hat{d}} + \frac{1}{4}\varepsilon^{kl}F^{\hat{m}}{}_{[\hat{a}}\varepsilon_{\hat{b}]\hat{m}\hat{n}\hat{d}\hat{e}}(\Gamma^{\hat{n}})_{\hat{\gamma}\hat{\delta}}N^{\hat{d}\hat{e}} - 2(\Sigma^{\hat{c}}{}_{[\hat{a}})_{\hat{\gamma}\hat{\delta}}F_{\hat{b}]\hat{c}}S^{kl}\\ &-\frac{1}{2}(\Sigma^{\hat{c}}{}_{[\hat{a}})_{\hat{\gamma}\hat{\delta}}N_{\hat{b}]\hat{d}}N^{\hat{c}\hat{d}} - \frac{1}{4}(\Sigma^{\hat{c}\hat{d}})_{\hat{\gamma}\hat{\delta}}N^{\hat{a}\hat{c}}N^{\hat{b}\hat{d}} - \frac{1}{8}(\Sigma_{\hat{a}\hat{b}})_{\hat{\gamma}\hat{\delta}}N^{\hat{c}\hat{d}}N_{\hat{c}\hat{d}}\\ &+\frac{1}{4}\varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}(\Gamma^{\hat{c}})_{\hat{\gamma}\hat{\delta}}N^{\hat{d}\hat{e}}S^{kl} + \frac{1}{2}(\Sigma_{\hat{a}\hat{b}})_{\hat{\gamma}\hat{\delta}}\varepsilon^{kl}S^{ij}S_{ij}\ .\end{aligned} \tag{2.55}$$

Considering the part of (2.54) which is symmetric in $\hat{\gamma}$ and $\hat{\delta}$ and also antisymmetric in k and l, we read off the expression for the Lorentz curvature $R_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}} = -\frac{1}{2}\varepsilon_{kl}(\Sigma^{\hat{c}\hat{d}})^{\hat{\gamma}\hat{\delta}}\Delta_{\hat{a}\hat{b}\hat{\gamma}\hat{\delta}}^{\ kl}$. The result is

$$\begin{aligned}R_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}} =& \frac{1}{4}\varepsilon^{\hat{c}\hat{d}}{}_{\hat{m}\hat{n}[\hat{a}}\mathcal{D}_{\hat{b}]}N^{\hat{m}\hat{n}} + \frac{\mathrm{i}}{8}(\Sigma^{\hat{c}\hat{d}})^{\hat{\gamma}\hat{\delta}}\mathcal{D}^k_{\hat{\gamma}}\mathcal{D}_{\hat{\delta}k}F_{\hat{a}\hat{b}} - \frac{1}{8}\delta^{[\hat{c}}_{\hat{a}}\delta^{\hat{d}]}_{\hat{b}}N^{\hat{m}\hat{n}}N_{\hat{m}\hat{n}}\\ &+\frac{1}{2}\delta^{[\hat{c}}_{[\hat{a}}N_{\hat{b}]\hat{m}}N^{\hat{d}]\hat{m}} - \frac{1}{4}N_{\hat{a}}{}^{[\hat{c}}N_{\hat{b}}{}^{\hat{d}]} - F_{\hat{a}}{}^{[\hat{c}}F_{\hat{b}}{}^{\hat{d}]} + \frac{1}{2}\delta^{[\hat{c}}_{\hat{a}}\delta^{\hat{d}]}_{\hat{b}}S^{ij}S_{ij}\ .\end{aligned} \tag{2.56}$$

Next, isolating the part of (2.54) which is proportional to $\varepsilon_{\hat{\gamma}\hat{\delta}}$ and symmetric in k, l, we can determine the SU(2)-curvature $R_{\hat{a}\hat{b}}{}^{kl} = -\frac{1}{4}\varepsilon^{\hat{\gamma}\hat{\delta}}\Delta_{\hat{a}\hat{b}\hat{\gamma}\hat{\delta}}^{\ kl}$. The result is

$$R_{\hat{a}\hat{b}}{}^{kl} = -\frac{\mathrm{i}}{8}\mathcal{D}^{\hat{\gamma}(k}\mathcal{D}^{l)}_{\hat{\gamma}}F_{\hat{a}\hat{b}}\ . \tag{2.57}$$

Equation (2.54) has allowed us to determine the curvatures. However it still contains some nontrivial information. Using the relations (2.53), (2.56) and (2.57), eq. (2.54) can be seen to reduce to

$$0 = -\frac{\mathrm{i}}{8}(\Gamma_{\hat{c}})^{\hat{\gamma}\hat{\delta}}\mathcal{D}^{(k}_{\hat{\gamma}}\mathcal{D}^{l)}_{\hat{\delta}}F_{\hat{a}\hat{b}} - \eta_{\hat{c}[\hat{a}}\mathcal{D}_{\hat{b}]}S^{kl} - \frac{1}{4}\varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}N^{\hat{d}\hat{e}}S^{kl}\ . \tag{2.58}$$

This implies

$$\mathcal{D}_{\hat{a}}S^{kl} = \frac{\mathrm{i}}{16}(\Gamma^{\hat{b}})^{\hat{\gamma}\hat{\delta}}\mathcal{D}^{(k}_{\hat{\gamma}}\mathcal{D}^{l)}_{\hat{\delta}}F_{\hat{a}\hat{b}} = -\frac{3\mathrm{i}}{16}\mathcal{D}^{\hat{\gamma}(k}\Xi_{\hat{a}\hat{\gamma}}{}^{l)} - \frac{\mathrm{i}}{8}(\Gamma_{\hat{a}})^{\hat{\gamma}\hat{\delta}}\mathcal{D}^{(k}_{\hat{\gamma}}\mathcal{F}_{\hat{\delta}}{}^{l)}\ . \tag{2.59}$$

Next, due to the identity

$$\mathcal{D}_{\hat{a}} = \frac{\mathrm{i}}{8}\varepsilon_{ij}(\Gamma_{\hat{a}})^{\hat{\alpha}\hat{\beta}}\mathcal{D}^i_{\hat{\alpha}}\mathcal{D}^j_{\hat{\beta}} + \frac{1}{8}\varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}N^{\hat{b}\hat{c}}M^{\hat{d}\hat{e}}\,, \tag{2.60}$$

and the dimension-3/2 constraint (2.15b) that determines $\mathcal{D}^i_{\hat\alpha} S^{kl}$, it also holds

$$\mathcal{D}_{\hat a} S^{kl} = -\frac{3\mathrm{i}}{16}(\Gamma_{\hat a})^{\hat\alpha\hat\beta}\mathcal{D}^{(k}_{\hat\alpha}\mathcal{F}_{\hat\beta}{}^{l)} - \frac{\mathrm{i}}{16}(\Gamma_{\hat a})^{\hat\alpha\hat\beta}\mathcal{D}^{(k}_{\hat\alpha}\mathcal{N}_{\hat\beta}{}^{l)} \ . \tag{2.61}$$

Now, requiring the compatibility of the equations (2.59) and (2.61), we generate the constraint

$$(\Gamma_{\hat a})^{\hat\alpha\hat\beta}\mathcal{D}^{(k}_{\hat\alpha}\mathcal{N}_{\hat\beta}{}^{l)} = -(\Gamma_{\hat a})^{\hat\alpha\hat\beta}\mathcal{D}^{(k}_{\hat\alpha}\mathcal{F}_{\hat\beta}{}^{l)} + 3\mathcal{D}^{\hat\gamma(k}\Xi_{\hat a\hat\gamma}{}^{l)} \ . \tag{2.62}$$

This turns out to be equivalent to (2.17a), since

$$\mathcal{D}^{\hat\alpha(k}\mathcal{N}_{\hat\alpha}{}^{l)} = \frac{1}{5}\{\mathcal{D}^{(k}_{\hat\alpha},\mathcal{D}^{l)}_{\hat\beta}\}N^{\hat\alpha\hat\beta} = \frac{4\mathrm{i}}{5}M_{\hat\alpha\hat\beta}N^{\hat\alpha\hat\beta} = 0 \ . \tag{2.63}$$

Similar considerations give $\mathcal{D}^{\hat\alpha(k}\mathcal{F}_{\hat\alpha}{}^{l)} = 0$.

Further analysis of equation (2.58) leads to another constraint, eq. (2.17d).

The Bianchi identity (2.53) is equivalent to $\varepsilon^{\hat a\hat b\hat c\hat d\hat e}\mathcal{D}_{\hat c}F_{\hat d\hat e} = 0$. The latter can be rewritten, with the aid of (2.60), as follows:

$$0 = \varepsilon^{\hat a\hat b\hat c\hat d\hat e}\Big(\,\mathrm{i}(\Gamma_{\hat c})^{\hat\alpha\hat\beta}\mathcal{D}^k_{\hat\alpha}\mathcal{D}_{\hat\beta k}F_{\hat d\hat e} + \varepsilon_{\hat c}{}^{\hat m\hat n\hat p\hat q}N_{\hat m\hat n}M_{\hat p\hat q}F_{\hat d\hat e}\Big)\,, \tag{2.64}$$

which, using (2.14), can be seen to be equivalent to equation (2.18).

Now, consider the Bianchi identity (2.52). Using the Lorentz curvature (2.56), eq. (2.52) turns out to be equivalent to

$$\begin{aligned}\mathcal{D}_{\hat a}N_{\hat b\hat c} = {} & \frac{\mathrm{i}}{8}\varepsilon_{\hat b\hat c\hat m\hat n\hat p}(\Sigma_{\hat a}{}^{\hat m})^{\hat\gamma\hat\delta}\mathcal{D}^k_{\hat\gamma}\mathcal{D}_{\hat\delta k}F^{\hat n\hat p} + \frac{\mathrm{i}}{12}\eta_{\hat a[\hat b}\varepsilon_{\hat c]\hat m\hat n\hat p\hat q}(\Sigma^{\hat m\hat n})^{\hat\gamma\hat\delta}\mathcal{D}^k_{\hat\gamma}\mathcal{D}_{\hat\delta k}F^{\hat p\hat q}\\ & -\frac{1}{12}\varepsilon_{\hat b\hat c\hat m\hat n\hat p}\big(4F_{\hat a}{}^{\hat m}F^{\hat n\hat p} + N_{\hat a}{}^{\hat m}N^{\hat n\hat p}\big) \ .\end{aligned} \tag{2.65}$$

The latter can rewritten as

$$\begin{aligned}\mathcal{D}_{\hat a}N_{\hat b\hat c} = {} & \frac{\mathrm{i}}{16}\varepsilon_{\hat a\hat b\hat c\hat d\hat e}\mathcal{D}^{\hat\delta k}W^{\hat d\hat e}{}_{\hat\delta k} - \frac{\mathrm{i}}{8}(\Gamma_{\hat a})^{\hat\gamma\hat\delta}\mathcal{D}^k_{\hat\gamma}W_{\hat b\hat c\hat\delta k} + \frac{\mathrm{i}}{2}(\Sigma_{\hat b\hat c})^{\hat\gamma\hat\delta}\mathcal{D}^k_{\hat\gamma}\Xi_{\hat a\hat\delta k} + \frac{\mathrm{i}}{8}\varepsilon_{\hat a\hat b\hat c\hat m\hat n}(\Gamma^{\hat m})^{\hat\gamma\hat\delta}\mathcal{D}^k_{\hat\gamma}\Xi^{\hat n}{}_{\hat\delta k}\\ & + \frac{\mathrm{i}}{8}\eta_{\hat a[\hat b}(\Gamma_{\hat c]})^{\hat\gamma\hat\delta}\mathcal{D}^k_{\hat\gamma}\mathcal{F}_{\hat\delta k} + \frac{\mathrm{i}}{8}\varepsilon_{\hat a\hat b\hat c\hat d\hat e}(\Sigma^{\hat d\hat e})^{\hat\gamma\hat\delta}\mathcal{D}^k_{\hat\gamma}\mathcal{F}_{\hat\rho k} - \frac{1}{12}\varepsilon_{\hat b\hat c\hat m\hat n\hat p}\big(4F_{\hat a}{}^{\hat m}F^{\hat n\hat p} + N_{\hat a}{}^{\hat m}N^{\hat n\hat p}\big).\end{aligned} \tag{2.66}$$

On the other hand, one can compute $\mathcal{D}_{\hat a}N_{\hat b\hat c}$ by using (2.60) and the dimension 3/2 Bianchi identity (2.16). Then one gets

$$\begin{aligned}\mathcal{D}_{\hat a}N_{\hat b\hat c} = {} & -\frac{\mathrm{i}}{8}(\Gamma_{\hat a})^{\hat\alpha\hat\beta}\mathcal{D}^k_{\hat\alpha}W_{\hat b\hat c\hat\beta k} + \frac{\mathrm{i}}{2}\eta_{\hat a[\hat b}\mathcal{D}^{\hat\alpha k}\Xi_{\hat c]\hat\alpha k} - \mathrm{i}(\Sigma_{\hat a[\hat b})^{\hat\alpha\hat\beta}\mathcal{D}^k_{\hat\alpha}\Xi_{\hat c]\hat\beta k}\\ & -\frac{\mathrm{i}}{16}\varepsilon_{\hat a\hat b\hat c\hat d\hat e}(\Sigma^{\hat d\hat e})^{\hat\alpha\hat\beta}\mathcal{D}^k_{\hat\alpha}\mathcal{N}_{\hat\beta k} + \frac{\mathrm{i}}{8}\eta_{\hat a[\hat b}(\Gamma_{\hat c]})^{\hat\alpha\hat\beta}\mathcal{D}^k_{\hat\alpha}\mathcal{N}_{\hat\beta k} - \frac{1}{2}\varepsilon_{\hat m\hat n\hat p\hat a[\hat b}N_{\hat c]}{}^{\hat m}N^{\hat n\hat p} \ .\end{aligned} \tag{2.67}$$

Requiring the equivalence of (2.66) and (2.67) and making use of (2.18), one obtains the constraints (2.17b) and (2.17c).

We have solved all Bianchi identities of dimension 2. Using the relations obtained, we can still simplify some of the results. Making use of (2.66) allows us to rewrite the Lorentz curvature (2.56) in the form:

$$
\begin{aligned}
R_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}} = \ & \frac{\mathrm{i}}{24}(\Sigma_{\hat{a}\hat{b}})^{\hat{\gamma}\hat{\delta}}\mathcal{D}^k_{\hat{\gamma}}\mathcal{D}_{\hat{\delta}k}F^{\hat{c}\hat{d}} + \frac{\mathrm{i}}{12}(\Sigma_{[\hat{a}}{}^{[\hat{c}})^{\hat{\gamma}\hat{\delta}}\mathcal{D}^k_{\hat{\gamma}}\mathcal{D}_{\hat{\delta}k}F_{\hat{b}]}{}^{\hat{d}]} + \frac{\mathrm{i}}{24}(\Sigma^{\hat{c}\hat{d}})^{\hat{\gamma}\hat{\delta}}\mathcal{D}^k_{\hat{\gamma}}\mathcal{D}_{\hat{\delta}k}F_{\hat{a}\hat{b}} \\
& -\frac{1}{3}F_{\hat{a}\hat{b}}F^{\hat{c}\hat{d}} - \frac{1}{3}F_{\hat{a}}{}^{[\hat{c}}F_{\hat{b}}{}^{\hat{d}]} - \frac{1}{12}N_{\hat{a}\hat{b}}N^{\hat{c}\hat{d}} - \frac{1}{12}N_{\hat{a}}{}^{[\hat{c}}N_{\hat{b}}{}^{\hat{d}]} + \frac{1}{2}\delta^{[\hat{c}}_{[\hat{a}}N_{\hat{b}]\hat{m}}N^{\hat{d}]\hat{m}} \\
& -\frac{1}{8}\delta^{[\hat{c}}_{\hat{a}}\delta^{\hat{d}]}_{\hat{b}}N^{\hat{m}\hat{n}}N_{\hat{m}\hat{n}} + \frac{1}{2}\delta^{[\hat{c}}_{\hat{a}}\delta^{\hat{d}]}_{\hat{b}}S^{ij}S_{ij} \ . \qquad (2.68)
\end{aligned}
$$

Next, using the equation

$$
\mathcal{D}^{\hat{\gamma}(k}W_{\hat{a}\hat{b}\hat{\gamma}}{}^{l)} = 3(\Sigma_{\hat{a}\hat{b}})^{\hat{\gamma}\hat{\delta}}\mathcal{D}^{(k}_{\hat{\gamma}}\mathcal{F}_{\hat{\delta}}{}^{l)} - 4(\Gamma_{[\hat{a}})^{\hat{\gamma}\hat{\delta}}\mathcal{D}^{(k}_{\hat{\gamma}}\Xi_{\hat{b}]\hat{\delta}}{}^{l)} + 12\mathrm{i}N_{\hat{a}\hat{b}}S^{kl} \, , \qquad (2.69)
$$

which follows from (2.17d), one can see that the SU(2)-curvature (2.57) can be rewritten as follows:

$$
R_{\hat{a}\hat{b}}{}^{kl} = \frac{3\mathrm{i}}{4}(\Gamma_{[\hat{a}})^{\hat{\gamma}\hat{\delta}}\mathcal{D}^{(k}_{\hat{\gamma}}\Xi_{\hat{b}]\hat{\delta}}{}^{l)} - \frac{\mathrm{i}}{4}(\Sigma_{\hat{a}\hat{b}})^{\hat{\gamma}\hat{\delta}}\mathcal{D}^{(k}_{\hat{\gamma}}\mathcal{F}_{\hat{\delta}}{}^{l)} + \frac{3}{2}N_{\hat{a}\hat{b}}S^{kl} \ . \qquad (2.70)
$$

Finally let us turn to the Bianchi identities of dimension 5/2 and 3. For dimension 5/2, there is only one nontrivial Bianchi identity. This is the identity (2.9a) with ($\hat{A} = \hat{a},\ \hat{B} = \hat{b},\ \hat{C} = \hat{c},\ \hat{D} = \underline{\hat{\delta}}$)

$$
0 = -\mathcal{D}_{\hat{a}}T_{\hat{b}\hat{c}l}{}^{\hat{\delta}} + T_{\hat{a}\hat{b}q}{}^{\hat{\rho}}T_{\hat{c}\hat{\rho}l}{}^{q\hat{\delta}} - \mathcal{D}_{\hat{b}}T_{\hat{c}\hat{a}l}{}^{\hat{\delta}} - T_{\hat{b}\hat{c}q}{}^{\hat{\rho}}T_{\hat{a}\hat{\rho}l}{}^{q\hat{\delta}} - \mathcal{D}_{\hat{c}}T_{\hat{a}\hat{b}l}{}^{\hat{\delta}} - T_{\hat{c}\hat{a}q}{}^{\hat{\rho}}T_{\hat{b}\hat{\rho}l}{}^{q\hat{\delta}} \ . \qquad (2.71)
$$

This equation can be seen to be satisfied identically provided the Bianchi identities of lower dimension hold. For dimension 3, there are no nontrivial Bianchi identities.

3. Projective superspace formalism

The projective superspace approach was originally formulated for rigid supersymmetric theories with eight supercharges in four space-time dimensions [3, 4], and later it was generalized to five [26] and six [27, 28] dimensions. Superconformal field theory in projective superspace has also been developed in four and five dimensions [29, 30].

As demonstrated in [1], the concept of projective supermultiplets can naturally be extended to the case of 5D $\mathcal{N} = 1$ supergravity. In this section, we first recall the definition of covariant projective multiplets in curved superspace, following [1]. After that we formulate a manifestly locally supersymmetric action principle.

To start with, it is instructive to recall the kinematical setup for projective superspace in the case of 5D $\mathcal{N} = 1$ supersymmetry. Let $\mathbb{R}^{5|8}$ denote the flat global superspace parametrized by coordinates $z^{\hat{A}} = (x^{\hat{a}}, \theta^{\hat{\alpha}}_i)$. The corresponding covariant derivatives $D_{\hat{A}} = (\partial_{\hat{a}}, D^i_{\hat{\alpha}})$ obey the algebra

$$
\{D^i_{\hat{\alpha}}\, ,\, D^j_{\hat{\beta}}\} = -2\mathrm{i}\,\varepsilon^{ij}\left((\Gamma^{\hat{c}})_{\hat{\alpha}\hat{\beta}}\,\partial_{\hat{c}} + \varepsilon_{\hat{\alpha}\hat{\beta}}\,Z\right), \qquad [D^i_{\hat{\alpha}}\, ,\, \partial_{\hat{b}}] = [D^i_{\hat{\alpha}}\, ,\, Z] = 0\, , \qquad (3.1)
$$

which follows from (2.12a)–(2.12c) by setting $S^{ij} = N_{\hat{a}\hat{b}} = F_{\hat{a}\hat{b}} = 0$. Making use of an isotwistor $u^+_i \in \mathbb{C}^2 \setminus \{0\}$ allows one to introduce a subset of strictly anti-commuting spinor covariant derivatives $D^+_{\hat{\alpha}} := u^+_i D^i_{\hat{\alpha}}$.

$$
\{D^+_{\hat{\alpha}}, D^+_{\hat{\beta}}\} = 0 \ . \qquad (3.2)
$$

Hence, one can define so-called *analytic* superfields $Q(z,u^+)$ constrained by $D^+_{\hat\alpha}Q=0$. Such a superfield $Q(z,u^+)$ is called a *projective supermultiplet*, if it is holomorphic (on an open subset of $\mathbb{C}^2\setminus\{0\}$) and a homogeneous function of u^+, $Q(z,c\,u^+)=c^n\,Q(z,u^+)$, with $c\in\mathbb{C}^*$. The isotwistor $u^+_i\in\mathbb{C}^2\setminus\{0\}$ appears to be defined modulo the equivalence relation $u^+_i\sim c\,u^+_i$, with $c\in\mathbb{C}^*$, since this is true for both the constraint $D^+_{\hat\alpha}Q=0$ and the superfield $Q(z,u^+)$ itself. As a result, the projective multiplets live in the projective superspace $\mathbb{R}^{4|8}\times\mathbb{C}P^1$.

3.1 Projective supermultiplets

In curved superspace, the isotwistor variables $u^+_i\in\mathbb{C}^2\setminus\{0\}$ are defined to be inert with respect to the local group SU(2) [1] (see also [2]). Instead of the anticommutation relation (3.2), the operators $\mathcal{D}^+_{\hat\alpha}:=u^+_i\,\mathcal{D}^i_{\hat\alpha}$ obey the following algebra:

$$\{\mathcal{D}^+_{\hat\alpha},\mathcal{D}^+_{\hat\beta}\}=-4\mathrm{i}\left(F_{\hat\alpha\hat\beta}+N_{\hat\alpha\hat\beta}\right)J^{++}+4\mathrm{i}\,S^{++}M_{\hat\alpha\hat\beta}\,, \tag{3.3}$$

where $J^{++}:=u^+_iu^+_j\,J^{ij}$ and $S^{++}:=u^+_iu^+_j\,S^{ij}$. Eq. (3.3) follows from (2.12a). Now, for the constraint $\mathcal{D}^+_{\hat\alpha}Q=0$ to be consistent, $Q(z,u^+)$ must be scalar with respect to the Lorentz group, $M_{\hat\alpha\hat\beta}Q=0$, and also possess special properties with respect to the group SU(2), that is, $J^{++}Q=0$. Let us define such multiplets, following [1].

A projective supermultiplet of weight n, $Q^{(n)}(z,u^+)$, is a scalar superfield that lives on $\mathcal{M}^{5|8}$, is holomorphic with respect to the isotwistor variables u^+_i on an open domain of $\mathbb{C}^2\setminus\{0\}$, and is characterized by the following conditions:

(i) it obeys the covariant analyticity constraint

$$\mathcal{D}^+_{\hat\alpha}Q^{(n)}=0\ ; \tag{3.4}$$

(ii) it is a homogeneous function of u^+ of degree n, that is,

$$Q^{(n)}(z,c\,u^+)\;=\;c^n\,Q^{(n)}(z,u^+)\,,\qquad c\in\mathbb{C}^*\ ; \tag{3.5}$$

(iii) infinitesimal gauge transformations (2.5) act on $Q^{(n)}$ as follows:

$$\begin{aligned}\delta Q^{(n)}&=\Big(K^{\hat C}\mathcal{D}_{\hat C}+K^{ij}J_{ij}\Big)Q^{(n)}\,,\\ K^{ij}J_{ij}Q^{(n)}&=-\frac{1}{(u^+u^-)}\Big(K^{++}D^{--}-n\,K^{+-}\Big)Q^{(n)}\,,\quad K^{\pm\pm}=K^{ij}\,u^\pm_iu^\pm_j\,,\end{aligned} \tag{3.6}$$

where

$$D^{--}=u^{-i}\frac{\partial}{\partial u^{+i}}\,,\qquad D^{++}=u^{+i}\frac{\partial}{\partial u^{-i}}\ . \tag{3.7}$$

The transformation law (3.6) involves an additional isotwistor, u^-_i, which is subject to the only condition $(u^+u^-)=u^{+i}u^-_i\neq0$, and is otherwise completely arbitrary. By construction, $Q^{(n)}$ is independent of u^-, i.e. $\partial Q^{(n)}/\partial u^{-i}=0$, and hence $D^{++}Q^{(n)}=0$. One can see that $\delta Q^{(n)}$ is also independent of the isotwistor u^-, $\partial(\delta Q^{(n)})/\partial u^{-i} = 0$, due to (3.5). It follows from (3.6)

$$J^{++}\, Q^{(n)} = 0\,, \qquad J^{++} \propto D^{++}\,, \tag{3.8}$$

and hence the covariant analyticity constraint (3.4) is indeed consistent.

The transformation law (3.6) is a generalization of that for superconformal projective supermultiplets in four and five dimensions [29, 30] and for projective supermultiplets in the 5D $\mathcal{N}=1$ anti-de Sitter superspace [2].

It should be pointed out that the transformation law (3.6) corresponds to the projective supermultiplets with zero central charge, $ZQ^{(n)} = 0$. Such off-shell multiplets are most interesting for applications, and our consideration will be restricted to their study. It is not difficult, however, to modify (3.6) in order to be applicable to the case of off-shell projective supermultiplets with an intrinsic zero central charge. The corresponding transformation law is [1]

$$\delta Q^{(n)} = \Big(K^{\hat C}\mathcal{D}_{\hat C} + K^{ij}J_{ij} + \tau Z\Big) Q^{(n)}\ . \tag{3.9}$$

As an example, we can consider an off-shell hypermultiplet with intrinsic central charge, which is described by $q^+(z,u^+) = u^+_i q^i(z)$. It is this realization[7] which is used in the component approaches of [10, 11]. In this realization, the hypermultiplet becomes on-shell provided $Zq^+ = 0$.

Given a projective multiplet $Q^{(n)}$, its complex conjugate is not covariantly analytic. However, similar to the flat four-dimensional case [14, 3] (see also [2]), one can introduce a generalized, analyticity-preserving conjugation, $Q^{(n)} \to \widetilde{Q}^{(n)}$, defined as

$$\widetilde{Q}^{(n)}(u^+) \equiv \bar{Q}^{(n)}\big(\overline{u^+} \to \widetilde{u}^+\big)\,, \qquad \widetilde{u}^+ = \mathrm{i}\,\sigma_2\, u^+\,, \tag{3.10}$$

with $\bar{Q}^{(n)}(\overline{u^+})$ the complex conjugate of $Q^{(n)}$. Its fundamental property is

$$\widetilde{\mathcal{D}^+_{\hat\alpha} Q^{(n)}} = (-1)^{\epsilon(Q^{(n)})}\, \mathcal{D}^{+\hat\alpha} \widetilde{Q}^{(n)}\ . \tag{3.11}$$

One can see that $\widetilde{\widetilde{Q}}^{(n)} = (-1)^n Q^{(n)}$, and therefore real supermultiplets can be consistently defined when n is even. In what follows, $\widetilde{Q}^{(n)}$ will be called the smile-conjugate of $Q^{(n)}$.

Examples of projective supermultiplets are given in [1], and the interested reader is referred to that paper for more details.

It follows from (2.15b) that S^{++} is a projective superfield of weight two,

$$\mathcal{D}^+_{\hat\alpha} S^{++} = 0\ . \tag{3.12}$$

3.2 Locally supersymmetric action

Let $\mathcal{L}^{++}$ be a real projective multiplet of weight two. Associated with $\mathcal{L}^{++}$ is the following functional

$$S(\mathcal{L}^{++}) = \frac{1}{6\pi}\oint (u^+ \mathrm{d}u^+) \int \mathrm{d}^5x\, \mathrm{d}^8\theta\, E\, \frac{\mathcal{L}^{++}}{(S^{++})^2}\,, \qquad E^{-1} = \mathrm{Ber}\,(E_{\hat A}{}^{\hat M})\ . \tag{3.13}$$

[7]This is a generalization of the Sohnius off-shell formulation for hypermultiplet [31].

We are going to show that S defines a locally supersymmetric action principle. This functional is obviously invariant under projective re-scalings $u_i^+ \to c\,u_i^+$. Moreover, it turns out to be invariant under infinitesimal gauge transformations (2.5) and (3.6). To prove the invariance under arbitrary supergravity gauge transformations, we first point out that $Q^{(-2)} := \mathcal{L}^{++}/(S^{++})^2$ is a projective multiplet of weight -2, since both $\mathcal{L}^{++}$ and S^{++} are projective multiplet of weight $+2$. For $Q^{(-2)}$ the second line in (3.6) implies

$$K^{ij} J_{ij}\, Q^{(-2)} = -\frac{1}{(u^+u^-)} D^{--}\Big(K^{++}Q^{(-2)}\Big)\ . \tag{3.14}$$

Next, since $K^{++}Q^{(-2)}$ has weight zero, it is easy to see

$$(u^+{\rm d}u^+)\, K^{ij} J_{ij}\, Q^{(-2)} = -{\rm d}t\, \frac{\rm d}{{\rm d}t}\, Q^{(-2)}\,, \tag{3.15}$$

with t the evolution parameter along the integration contour in (3.13). Since the integration contour is closed, the SU(2)-part of the transformation (3.6) does not contribute to the variation of the action (3.13). To complete the proof, it remains to take into the account the fact that $Q^{(-2)}$ is a Lorentz scalar.

Introduce the following fourth-order operator[8]

$$\Delta^{(+4)} = (\mathcal{D}^+)^4 - \frac{5}{12}{\rm i}\, S^{++}\, (\mathcal{D}^+)^2 + 3(S^{++})^2\,, \tag{3.16}$$

where

$$(\mathcal{D}^+)^4 := -\frac{1}{96}\varepsilon^{\hat\alpha\hat\beta\hat\gamma\hat\delta}\mathcal{D}^+_{\hat\alpha}\mathcal{D}^+_{\hat\beta}\mathcal{D}^+_{\hat\gamma}\mathcal{D}^+_{\hat\delta}\,, \qquad (\mathcal{D}^+)^2 := \mathcal{D}^{+\hat\alpha}\mathcal{D}^+_{\hat\alpha}\ . \tag{3.17}$$

Its crucial property is that the superfield $Q^{(n)}$ defined by

$$Q^{(n)}(z,u^+) := \Delta^{(+4)} U^{(n-4)}(z,u^+)\,, \tag{3.18}$$

is a weight-n projective multiplet,

$$\mathcal{D}^+_{\hat\alpha} Q^{(n)} = 0\,, \tag{3.19}$$

for any *unconstrained* scalar superfield $U^{(n-4)}(z,u^+)$ that lives on $\mathcal{M}^{5|8}$, is holomorphic with respect to the isotwistor variables u_i^+ on an open domain of $\mathbb{C}^2 \setminus \{0\}$, and is characterized by the following conditions:
(i) it is a homogeneous function of u^+ of degree $n-4$, that is,

$$U^{(n-4)}(z,c\,u^+) \;=\; c^{n-4}\, U^{(n-4)}(z,u^+)\,, \qquad c\in\mathbb{C}^*\ ; \tag{3.20}$$

(iii) infinitesimal gauge transformations (2.5) act on $U^{(n-4)}$ as follows:

$$\begin{aligned} \delta U^{(n-4)} &= \Big(K^{\hat C}\mathcal{D}_{\hat C} + K^{ij}J_{ij}\Big)U^{(n-4)}\,, \\ K^{ij}J_{ij}\, U^{(n-4)} &= -\frac{1}{(u^+u^-)}\Big(K^{++}D^{--} - (n-4)\,K^{+-}\Big)U^{(n-4)}\ . \end{aligned} \tag{3.21}$$

[8]This operator was introduced in the case of 5D $\mathcal{N}=1$ anti-de Sitter supersymmetry in [2].

We will call $U^{(n-4)}(z, c\,u^+)$ a *projective prepotential* for $Q^{(n)}$.

The fourth-order operator (3.16) is analogous to the chiral projector in 4D $\mathcal{N} = 1$ supergravity [32].

Let $U^{(-2)}$ be a projective prepotential for the Lagrangian $\mathcal{L}^{++}$ in (3.13). Representing

$$U^{(-2)} = \frac{1}{3(S^{++})^2}\Big\{\mathcal{L}^{++} - (\mathcal{D}^+)^4 U^{(-2)} + \frac{5}{12}\mathrm{i}\, S^{++}\,(\mathcal{D}^+)^2 U^{(-2)}\Big\}\,, \tag{3.22}$$

we obtain

$$\frac{1}{2\pi}\oint (u^+\mathrm{d}u^+) \int \mathrm{d}^5x\,\mathrm{d}^8\theta\, E\, U^{(-2)} = \frac{1}{6\pi}\oint (u^+\mathrm{d}u^+) \int \mathrm{d}^5x\,\mathrm{d}^8\theta\, E\, \frac{\mathcal{L}^{++}}{(S^{++})^2}\,. \tag{3.23}$$

One can see that the derivative terms in (3.22) do not contribute to the integral in (3.23), as a consequence of the anti-commutation relations (2.12a)–(2.12c).

Our action (3.13) can be compared with the chiral action in 4D $\mathcal{N} = 1$ supergravity [32, 33] (see also [23, 24] for reviews).

In the case of flat superspace, one can not make use of (3.23). Instead, here one can apply the following relations

$$\begin{aligned}\frac{1}{2\pi}\oint (u^+\mathrm{d}u^+) \int \mathrm{d}^5x\,\mathrm{d}^8\theta\, U^{(-2)} &= \frac{1}{2\pi}\oint \frac{(u^+\mathrm{d}u^+)}{(u^+u^-)^4} \int \mathrm{d}^5x\,(D^-)^4(D^+)^4 U^{(-2)}\Big|_{\theta=0}\\ &= \frac{1}{2\pi}\oint \frac{(u^+\mathrm{d}u^+)}{(u^+u^-)^4} \int \mathrm{d}^5x\,(D^-)^4 L^{++}\Big|_{\theta=0}\,,\end{aligned} \tag{3.24}$$

with $L^{++} := (D^+)^4 U^{(-2)}$ the flat-superspace Lagrangan. Here

$$(D^-)^4 := -\frac{1}{96}\varepsilon^{\hat\alpha\hat\beta\hat\gamma\hat\delta} D^-_{\hat\alpha} D^-_{\hat\beta} D^-_{\hat\gamma} D^-_{\hat\delta}\,, \qquad D^-_{\hat\alpha} := u^-_i D^i_{\hat\alpha}\,. \tag{3.25}$$

The expression in the second line of (3.24) is the rigid supersymmetric action in 5D $\mathcal{N} = 1$ projective superspace [29]. The latter is a natural generalization of the 4D $\mathcal{N} = 2$ projective-superspace action originally given in [3] and reformulated in a projective-invariant form in [34]. This action can be seen to be invariant under arbitrary transformations of the form:

$$(u_i{}^-\,,\, u_i{}^+) \;\to\; (u_i{}^-\,,\, u_i{}^+)\, R\,, \qquad R = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})\,. \tag{3.26}$$

The same invariance obviously holds for the curved-superspace action (3.13), for it is explicitly independent of u^-.

Projective invariance (3.26) is an obvious property of the manifestly locally supersymmetric action (3.13). As shown in section 5, it becomes a powerful constructive principle when one is interested in reducing the action to components in the Wess-Zumino gauge.

4. Wess-Zumino gauge

In this section we elaborate the Wess-Zumino gauge for the 5D minimal supergravity multiplet, which was used in [1]. Our consideration will be similar to that originally given, many years ago, for 4D $\mathcal{N}=1$ supergravity [32, 35, 36] and then presented in a universally applicable form in [23].

Given a superfield $U(z)=U(x,\theta)$, it is standard to denote as $U|$ its θ-independent component, $U|:=U(x,\theta=0)$. The Wess-Zumino (WZ) gauge for 5D $\mathcal{N}=1$ supergravity is defined by

$$\mathcal{D}_{\hat{a}}\big| = \nabla_{\hat{a}} + \Psi_{\hat{a}}{}^{\hat{\gamma}}_k(x)\mathcal{D}^k_{\hat{\gamma}}\big| + \phi_{\hat{a}}{}^{kl}(x)J_{kl} + \mathcal{V}_{\hat{a}}(x)Z\,, \qquad \mathcal{D}^i_{\hat{\alpha}}\big| = \frac{\partial}{\partial\theta^{\hat{\alpha}}_i}\,. \tag{4.1}$$

Here $\nabla_{\hat{a}}$ denotes the space-time covariant derivatives,

$$\nabla_{\hat{a}} = e_{\hat{a}} + \omega_{\hat{a}}\,, \qquad e_{\hat{a}} = e_{\hat{a}}{}^{\hat{m}}(x)\,\partial_{\hat{m}}\,, \qquad \omega_{\hat{a}} = \frac{1}{2}\,\omega_{\hat{a}}{}^{\hat{b}\hat{c}}(x)\,M_{\hat{b}\hat{c}} = \omega_{\hat{a}}{}^{\hat{\beta}\hat{\gamma}}(x)\,M_{\hat{\beta}\hat{\gamma}}\,, \tag{4.2}$$

with $e_{\hat{a}}{}^{\hat{m}}$ the component inverse vielbein, and $\omega_{\hat{a}}{}^{\hat{b}\hat{c}}$ the Lorentz connection. The operators $\nabla_{\hat{a}}$ obey commutation relations of the form

$$\big[\nabla_{\hat{a}},\nabla_{\hat{b}}\big] = \mathcal{T}_{\hat{a}\hat{b}}{}^{\hat{c}}(x)\,\nabla_{\hat{c}} + \frac{1}{2}\mathcal{R}_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}}(x)M_{\hat{c}\hat{d}}\,, \tag{4.3}$$

with $\mathcal{T}_{\hat{a}\hat{b}}{}^{\hat{c}}$ the torsion, and $\mathcal{R}_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}}$ the curvature. Next, $\Psi_{\hat{a}}{}^{\hat{\gamma}}_k$ is the component gravitino, while $\phi_{\hat{a}}{}^{kl}=\Phi_{\hat{a}}{}^{kl}|$ and $\mathcal{V}_{\hat{a}}=V_{\hat{a}}|$ are the component SU(2) and central-charge gauge fields, respectively. In addition to the geometric fields present in (4.1), the supergravity multiplet includes some additional component fields which can be chosen as follows: $S_{ij}\big|$, $N_{\hat{a}\hat{b}}\big|$, $\mathcal{D}^j_{\hat{\alpha}}S_{ij}\big|$ and $\mathcal{D}^{\hat{\alpha}i}\mathcal{D}^j_{\hat{\alpha}}S_{ij}\big|$. All these fields, which survive in the WZ gauge, constitute the 5D minimal supergravity multiplet [20].

Making use of (4.1) one can readily obtain

$$\begin{aligned}
[\mathcal{D}_{\hat{a}},\mathcal{D}_{\hat{b}}]\big| = {} & [\nabla_{\hat{a}},\nabla_{\hat{b}}] - 2\Psi_{[\hat{a}}{}^{\hat{\gamma}}_k\,[\mathcal{D}_{\hat{b}]},\mathcal{D}^k_{\hat{\gamma}}]\big| + \Psi_{\hat{a}}{}^{\hat{\gamma}}_k\Psi_{\hat{b}}{}^{\hat{\delta}}_l\,\{\mathcal{D}^k_{\hat{\gamma}},\mathcal{D}^l_{\hat{\delta}}\}\big| + 2(\nabla_{[\hat{a}}\mathcal{V}_{\hat{b}]})Z \\
& +2\Big(\nabla_{[\hat{a}}\Psi_{\hat{b}]}{}^{\hat{\gamma}}_k - \phi_{[\hat{a}k}{}^l\Psi_{\hat{b}]}{}^{\hat{\gamma}}_l\Big)\mathcal{D}^k_{\hat{\gamma}}\big| + 2\Big(\nabla_{[\hat{a}}\phi_{\hat{b}]}{}^{kl} + \phi_{[\hat{a}}{}^k{}_j\phi_{\hat{b}]}{}^{jl}\Big)J_{kl}\,.
\end{aligned} \tag{4.4}$$

This relation can be simplified considerably by evaluating the (anti-)commutators $[\mathcal{D}_{\hat{a}},\mathcal{D}_{\hat{b}}]$, $[\mathcal{D}_{\hat{b}},\mathcal{D}^k_{\hat{\gamma}}]$ and $\{\mathcal{D}^k_{\hat{\gamma}},\mathcal{D}^l_{\hat{\delta}}\}$ with the aid of (2.12a)–(2.12c). As a result, eq. (4.4) can be seen to be equivalent to the following relations:

$$\mathcal{T}_{\hat{a}\hat{b}}{}^{\hat{c}} = -2\mathrm{i}\Psi_{\hat{a}}{}^{\hat{\gamma}k}(\Gamma^{\hat{c}})_{\hat{\gamma}\hat{\delta}}\Psi_{\hat{b}}{}^{\hat{\delta}}_k\,, \tag{4.5a}$$

$$T_{\hat{a}\hat{b}}{}^{\hat{\gamma}}_k\big| = \frac{\mathrm{i}}{2}\mathcal{D}^{\hat{\gamma}}_kF_{\hat{a}\hat{b}}\big| = 2\nabla_{[\hat{a}}\Psi_{\hat{b}]}{}^{\hat{\gamma}}_k - 2\phi_{[\hat{a}k}{}^j\Psi_{\hat{b}]}{}^{\hat{\gamma}}_j - \mathcal{T}_{\hat{a}\hat{b}}{}^{\hat{c}}\Psi_{\hat{c}}{}^{\hat{\gamma}}_k - 2\Psi_{[\hat{a}}{}^{\hat{\beta}}_jT_{\hat{b}]}{}^j_{\hat{\beta}}{}^{\hat{\gamma}}_k\big|\,, \tag{4.5b}$$

as well as

$$R_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}}\big| = \mathcal{R}_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}} - 2\Psi_{[\hat{a}}{}^{\hat{\gamma}}_kR_{\hat{b}]}{}^k_{\hat{\gamma}}{}^{\hat{c}\hat{d}}\big| + \Psi_{\hat{a}}{}^{\hat{\gamma}}_k\Psi_{\hat{b}}{}^{\hat{\delta}}_lR^{kl}_{\hat{\gamma}\hat{\delta}}{}^{\hat{c}\hat{d}}\big|\,, \tag{4.6a}$$

$$F_{\hat{a}\hat{b}}\big| = 2(\nabla_{[\hat{a}}\mathcal{V}_{\hat{b}]}) + 2\mathrm{i}\Psi_{\hat{a}}{}^{\hat{\gamma}k}\Psi_{\hat{b}}{}^{\hat{\delta}}_k\mathcal{V}_{\hat{\gamma}\hat{\delta}} - 2\mathrm{i}\Psi_{\hat{a}}{}^{\hat{\gamma}}_k\Psi_{\hat{b}}{}^k_{\hat{\gamma}}\,, \tag{4.6b}$$

$$\begin{aligned}
R_{\hat{a}\hat{b}}{}^{ij}\big| = {} & 2\nabla_{[\hat{a}}\phi_{\hat{b}]}{}^{ij} + 2\phi_{[\hat{a}}{}^i{}_k\phi_{\hat{b}]}{}^{kj} \\
& -2\Psi_{[\hat{a}}{}^{\hat{\gamma}}_kR_{\hat{b}]}{}^k_{\hat{\gamma}}{}^{ij}\big| + \Psi_{\hat{a}}{}^{\hat{\gamma}}_k\Psi_{\hat{b}}{}^{\hat{\delta}}_lR^{kl}_{\hat{\gamma}\hat{\delta}}{}^{ij}\big| + 2\mathrm{i}\Psi_{\hat{a}}{}^{\hat{\gamma}k}\Psi_{\hat{b}}{}^{\hat{\delta}}_k\phi_{\hat{\gamma}\hat{\delta}}{}^{ij}\,.
\end{aligned} \tag{4.6c}$$

Eq. (4.5a) determines the space-time torsion in terms of the gravitino. Eq. (4.5b) constitutes a locally supersymmetric version of the gravitino field strength. Finally, eqs. (4.6a)–(4.6c) express the leading components of the superspace curvature tensors in terms of the component fields. Equations (4.5a) and (4.5b) will frequently be used in section 5. The space-time torsion (4.5a) will be especially important for the considerations in section 5, for it occurs in the rule for integration by parts:

$$\int \mathrm{d}^5x\, e\, \nabla_{\hat a} U^{\hat a} = \int \mathrm{d}^5x\, e\, \mathcal{T}_{\hat a \hat b}{}^{\hat b}\, U^{\hat a}\,, \qquad e^{-1} = \det\left(e_{\hat a}{}^{\hat m}\right)\,. \tag{4.7}$$

In the WZ gauge, the supergravity gauge fredom (2.5) reduces to those transformations which preserve the WZ gauge. This is equivalent to the requirement

$$0 = \delta \mathcal{D}^i_{\hat\alpha}\big| = -\Big[K^{\hat\beta}_j \mathcal{D}^j_{\hat\beta} + K^{\hat b}\mathcal{D}_{\hat b} + K^{\hat\beta\hat\gamma}M_{\hat\beta\hat\gamma} + K^{jk}J_{jk} + \tau Z, \mathcal{D}^i_{\hat\alpha}\Big]\Big|\,. \tag{4.8}$$

It implies the following restrictions on the transformation parameters:

$$\begin{aligned}
&\mathcal{D}^i_{\hat\alpha}K^{\hat\beta}_j\big| = K^{\hat c}\big|\,T_{\hat c}{}^{i}_{\hat\alpha j}{}^{\hat\beta}\big| + \delta^i_j\,K_{\hat\alpha}{}^{\hat\beta}\big| + \delta^{\hat\beta}_{\hat\alpha}K^i{}_j\big|\,, && \mathcal{D}^i_{\hat\alpha}K^{\hat b}\big| = -2\mathrm{i}\,(\Gamma^{\hat b})_{\hat\alpha\hat\gamma}\,K^{\hat\gamma i}\big|\,,\\
&\mathcal{D}^i_{\hat\alpha}K^{\hat\beta\hat\gamma}\big| = K^{\hat C}\big|\,R_{\hat C}{}^{i}_{\hat\alpha}{}^{\hat\beta\hat\gamma}\big|\,, \quad \mathcal{D}^i_{\hat\alpha}K^{jk}\big| = K^{\hat C}\big|\,R_{\hat C}{}^{i}_{\hat\alpha}{}^{jk}\big|\,, && \mathcal{D}^i_{\hat\alpha}\tau\big| = -2\mathrm{i}\,K^i_{\hat\alpha}\big|\,.
\end{aligned} \tag{4.9}$$

In the WZ gauge, the transformation laws of the gauge fields can be derived from

$$\begin{aligned}
\delta\mathcal{D}_{\hat a}\big| &= \delta\nabla_{\hat a} + \delta\Psi_{\hat a}{}^{\hat\beta}_j\mathcal{D}^j_{\hat\beta}\big| + \delta\phi_{\hat a}{}^{kl}J_{kl} + \delta\mathcal{V}_{\hat a}Z\\
&= -\Big[K^{\hat B}\mathcal{D}_{\hat B} + K^{\hat\beta\hat\gamma}M_{\hat\beta\hat\gamma} + K^{jk}J_{jk} + \tau Z, \mathcal{D}_{\hat a}\Big]\Big|\,.
\end{aligned} \tag{4.10}$$

Some computations lead to

$$\delta e_{\hat a}{}^{\hat m} = \Big(\nabla_{\hat a}K^{\hat b}\big| - 2\mathrm{i}\Psi_{\hat a}{}^{\hat\alpha}_k(\Gamma^{\hat b})_{\hat\alpha\hat\beta}K^{\hat\beta k}\big| - K_{\hat a}{}^{\hat b}\big|\Big)e_{\hat b}{}^{\hat m}\,, \tag{4.11a}$$

$$\begin{aligned}
\delta\omega_{\hat a}{}^{\hat\beta\hat\gamma} &= \Big(\nabla_{\hat a}K^{\hat b}\big| - 2\mathrm{i}\Psi_{\hat a}{}^{\hat\alpha}_k(\Gamma^{\hat b})_{\hat\alpha\hat\delta}K^{\hat\delta k}\big| - K_{\hat a}{}^{\hat b}\big|\Big)\omega_{\hat b}{}^{\hat\beta\hat\gamma} + \nabla_{\hat a}K^{\hat\gamma\hat\delta}\big|\\
&\quad +\Psi_{\hat a}{}^{\hat\beta}_jK^{\hat C}\big|R_{\hat C}{}^{j}_{\hat\beta}{}^{\hat\gamma\hat\delta}\big| - K^{\hat B}\big|R_{\hat B\hat a}{}^{\hat\gamma\hat\delta}\big|\,,
\end{aligned} \tag{4.11b}$$

$$\begin{aligned}
\delta\Psi_{\hat a}{}^{\hat\beta}_j &= \nabla_{\hat a}K^{\hat\beta}_j\big| - \phi_{\hat a j}{}^kK^{\hat\beta}_k\big| - 2\mathrm{i}\Psi_{\hat a}{}^{\hat\gamma}_j(\Gamma^{\hat c})_{\hat\gamma\hat\delta}\Psi_{\hat c}{}^{\hat\beta}_kK^{\hat\delta k}\big| + K^{\hat\gamma}_k\big|T_{\hat a}{}^{k}_{\hat\gamma j}{}^{\hat\beta}\big| + \nabla_{\hat a}K^{\hat c}\big|\Psi_{\hat c}{}^{\hat\beta}_j\\
&\quad +\Psi_{\hat a}{}^{\hat\gamma}_kK^{\hat c}\big|T_{\hat c}{}^{k}_{\hat\gamma j}{}^{\hat\beta}\big| - K^{\hat c}\big|T_{\hat c\hat a}{}^{\hat\beta}_j\big| - K_{\hat a}{}^{\hat c}\big|\Psi_{\hat c}{}^{\hat\beta}_j + \Psi_{\hat a}{}^{\hat\gamma}_jK_{\hat\gamma}{}^{\hat\beta}\big| + \Psi_{\hat a}{}^{\hat\gamma}_kK^k{}_j\big|\,,
\end{aligned} \tag{4.11c}$$

$$\begin{aligned}
\delta\phi_{\hat a}{}^{jk} &= \Big(\nabla_{\hat a}K^{\hat b}\big| - 2\mathrm{i}\Psi_{\hat a}{}^{\hat\beta}_l(\Gamma^{\hat b})_{\hat\beta\hat\gamma}K^{\hat\gamma l}\big| - K_{\hat a}{}^{\hat b}\big|\Big)\phi_{\hat b}{}^{jk} + \nabla_{\hat a}K^{jk}\big| + 2\phi_{\hat a}{}^{(j}{}_lK^{k)l}\big|\\
&\quad +K^{\hat\beta}_l\big|R_{\hat a}{}^{l}_{\hat\beta}{}^{jk}\big| + K^{\hat b}\big|R_{\hat a\hat b}{}^{jk}\big| + \Psi_{\hat a}{}^{\hat\beta}_lK^{\hat C}\big|R_{\hat C}{}^{l}_{\hat\beta}{}^{jk}\big|\,,
\end{aligned} \tag{4.11d}$$

$$\delta\mathcal{V}_{\hat a} = \Big(\nabla_{\hat a}K^{\hat b}\big| - 2\mathrm{i}\Psi_{\hat a}{}^{\hat\beta}_j(\Gamma^{\hat b})_{\hat\beta\hat\gamma}K^{\hat\gamma j}\big| - K_{\hat a}{}^{\hat b}\big|\Big)\mathcal{V}_{\hat b} + \nabla_{\hat a}\tau\big| + K^{\hat b}\big|F_{\hat a\hat b}\big| - 2\mathrm{i}\Psi_{\hat a}{}^{\hat\beta}_jK^j_{\hat\beta}\big|\,. \tag{4.11e}$$

5. Action principle in the Wess-Zumino gauge

Our goal in this section is to reduce the locally supersymmetric action (3.13) to components in the WZ gauge. Using considerations based on eqs. (3.23), (3.24), (4.1) and $E| = e$, one can argue that in the WZ gauge it holds

$$S(\mathcal{L}^{++}) = S_0\ +\ \ldots\,, \qquad S_0 = \frac{1}{2\pi}\oint\frac{(u^+\mathrm{d}u^+)}{(u^+u^-)^4}\int\mathrm{d}^5x\,e\,(\mathcal{D}^-)^4\mathcal{L}^{++}(z,u^+)\Big|\,, \tag{5.1}$$

where

$$\mathcal{D}^-_{\hat\alpha} := u^-_i \mathcal{D}^i_{\hat\alpha}\,, \qquad (\mathcal{D}^-)^4 := -\frac{1}{96}\varepsilon^{\hat\alpha\hat\beta\hat\gamma\hat\delta}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta}\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\delta}\,, \tag{5.2}$$

and the dots in the expression for $S(\mathcal{L}^{++})$ in (5.1) denote all the terms with at most three spinor derivatives hitting $\mathcal{L}^{++}$.

By construction, the action (3.13) is invariant under arbitrary projective transformations (3.26). It is remarkable that the requirement of projective invariance allows one to uniquely restore the action in the WZ gauge by making use of S_0, as given in (5.1), as the only input. Let us start by presenting the result of explicit calculations announced in [1]:

$$\begin{aligned} S(\mathcal{L}^{++}) = \frac{1}{2\pi}\oint_\gamma \frac{(u^+\mathrm{d}u^+)}{(u^+u^-)^4}\int \mathrm{d}^5x e\Big[&(\mathcal{D}^-)^4 + \frac{\mathrm{i}}{4}\Psi^{\hat\alpha\hat\beta\hat\gamma-}\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta} - \frac{25}{24}\mathrm{i}S^{--}(\mathcal{D}^-)^2 \\ &-2(\Sigma^{\hat a\hat b})_{\hat\beta}{}^{\hat\gamma}\Psi_{\hat a}{}^{\hat\beta-}\Psi_{\hat b}{}^{\hat\delta-}\mathcal{D}_{[\hat\gamma}{}^-\mathcal{D}_{\hat\delta]}{}^- - \frac{\mathrm{i}}{4}\phi^{\hat\alpha\hat\beta--}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta} + 4(\Sigma^{\hat a\hat b})^{\hat\alpha}{}_{\hat\gamma}\phi_{[\hat a}{}^{--}\Psi_{\hat b]}{}^{\hat\gamma-}\mathcal{D}^-_{\hat\alpha} \\ &-4\,\Psi^{\hat\alpha\hat\beta}{}_{\hat\beta}^{-}S^{--}\mathcal{D}^-_{\hat\alpha} + 2\mathrm{i}\,\varepsilon^{\hat a\hat b\hat c\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}\Psi_{\hat a}{}^{\hat\alpha-}\Psi_{\hat b}{}^{\hat\beta-}\Psi_{\hat c}{}^{\hat\gamma-}\mathcal{D}^-_{\hat\gamma} + 18S^{--}S^{--} \\ &-6\mathrm{i}\,\varepsilon^{\hat a\hat b\hat c\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}\Psi_{\hat a}{}^{\hat\alpha-}\Psi_{\hat b}{}^{\hat\beta-}\phi_{\hat c}{}^{--} + 18\mathrm{i}\,(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}\Psi_{\hat a}{}^{\hat\alpha-}\Psi_{\hat b}{}^{\hat\beta-}S^{--}\Big]\mathcal{L}^{++}\Big|, \end{aligned} \tag{5.3}$$

where $(\mathcal{D}^-)^2 := \mathcal{D}^{\hat\alpha-}\mathcal{D}^-_{\hat\alpha}$, $S^{--} := S^{ij}u^-_i u^-_j$, $\phi_{\hat a}{}^{--} := \phi_{\hat a}{}^{ij}u^-_i u^-_j$ and $\Psi_{\hat a}{}^{\hat\beta-} := \Psi_{\hat a}{}^{\hat\beta i}u^-_i$.

The remainder of this section is devoted to the derivation of (5.3). Conceptually, our approach below is quite simple. We start by computing the variation of S_0 (5.1) under an infinitesimal projective transformation (3.26), and then iteratively add new terms to the action in order to cancel out all non-zero contributions to the variation, insuring projective invariance in the end. Technically, the calculation turns out to be quite long.

In the following, we will use the condensed notation:

$$\mathrm{d}\mu^{++} := \frac{1}{2\pi}\frac{(u^+\mathrm{d}u^+)}{(u^+u^-)^4} = -\frac{1}{2\pi}\frac{(\dot u^+u^+)}{(u^+u^-)^4}\,\mathrm{d}t\,, \tag{5.4}$$

where we have denoted $\dot f := \mathrm{d}f(t)/\mathrm{d}t$, for a function $f(t)$. Here t is the time parameter along the closed integration contour $\gamma = \{u^+_i(t)\}$ in the isotwistor space which occurs in (5.1). In the integrand of (5.1), the isotwistor u^-_i is chosen to be constant (i.e. time-independent) and subject to the condition that $u^+(t)$ and u^- form a linearly independent basis at each point of the contour γ, that is $(u^+u^-) \neq 0$.

Concerning the projective transformations (3.26), it is obvious that S_0 (5.1) is invariant under arbitrary scale transformations $u^+_i(t) \to c(t)\,u^+_i(t)$, with $c(t) \neq 0$. The iterative contributions to S should be chosen to automatically respect this invariance. It is thus only necessary to analyse projective transformations of u^- of the form

$$u^-_i \;\to\; \tilde u^-_i \;=\; a(t)\,u^-_i + b(t)\,u^+_i(t)\,, \qquad a(t) \neq 0\,. \tag{5.5}$$

Since both u^- and $\tilde u^-$ should be time independent, the coefficients should obey the equations:

$$\dot a = b\,\frac{(\dot u^+u^+)}{(u^+u^-)}\,, \qquad \dot b = -b\,\frac{(\dot u^+u^-)}{(u^+u^-)}\,. \tag{5.6}$$

As is obvious, the functional S_0 (5.1) is invariant under arbitrary scale transformations $u_i^- \to a(t)\, u_i^-$, with $a \neq 0$. The other contributions to S, which we are going to determine, should be chosen to automatically respect this invariance. Therefore, it only remains to analyse infinitesimal transformations of the form $\delta u_i^- = b(t) u_i^+$, with $b(t)$ obeying the differential equation (5.6). This transformation induces the following variation of S_0:

$$\begin{aligned}\delta S_0 &= \oint \mathrm{d}\mu^{++} \int \mathrm{d}^5x\, e \left[\delta(\mathcal{D}^-)^4\right] \mathcal{L}^{++}\Big| \\ &= -\frac{\varepsilon^{\hat\alpha\hat\beta\hat\gamma\hat\delta}}{96} \oint \mathrm{d}\mu^{++}\, b \int \mathrm{d}^5x\, e \Big[\, 3\,\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta}\{\mathcal{D}^+_{\hat\gamma}, \mathcal{D}^-_{\hat\delta}\} + 2\,\mathcal{D}^-_{\hat\alpha}\{\mathcal{D}^+_{\hat\beta}, \mathcal{D}^-_{\hat\gamma}\}\mathcal{D}^-_{\hat\delta} \\ &\qquad\qquad + \{\mathcal{D}^+_{\hat\alpha}, \mathcal{D}^-_{\hat\beta}\}\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\delta}\,\Big]\mathcal{L}^{++}\Big| \,.\end{aligned} \tag{5.7}$$

First of all, this variation has to be transformed.

Using the completeness relation

$$(u^+u^-)\,\delta^i_j = u^{+i}u^-_j - u^{-i}u^+_j \,, \tag{5.8}$$

the (anti-)commutation relations (2.4), (2.12a) and (2.12b) can be seen to imply

$$[J_{kl}, \mathcal{D}^\pm_{\hat\alpha}] = \frac{1}{(u^+u^-)}\Big[u^\pm_{(k}u^-_{l)}\mathcal{D}^+_{\hat\alpha} - u^\pm_{(k}u^+_{l)}\mathcal{D}^-_{\hat\alpha}\Big]\,, \tag{5.9a}$$

$$\{\mathcal{D}^+_{\hat\alpha}, \mathcal{D}^-_{\hat\beta}\} = 2\mathrm{i}(u^+u^-)\mathcal{D}_{\hat\alpha\hat\beta} + R^{+}_{\hat\alpha}{}^{-}_{\hat\beta}{}^{kl}J_{kl} + R^{+}_{\hat\alpha}{}^{-}_{\hat\beta}{}^{\hat\gamma\hat\delta}M_{\hat\gamma\hat\delta} + 2\mathrm{i}(u^+u^-)\varepsilon_{\hat\alpha\hat\beta}Z\,, \tag{5.9b}$$

$$[\mathcal{D}_{\hat\alpha\hat\beta}, \mathcal{D}^\pm_{\hat\gamma}] = \frac{1}{(u^+u^-)}T_{\hat\alpha\hat\beta}{}^{\pm}_{\hat\gamma}{}^{\hat\delta-}\mathcal{D}^+_{\hat\delta} - \frac{1}{(u^+u^-)}T_{\hat\alpha\hat\beta}{}^{\pm}_{\hat\gamma}{}^{\hat\delta+}\mathcal{D}^-_{\hat\delta} + R_{\hat\alpha\hat\beta}{}^{\pm}_{\hat\gamma}{}^{lp}J_{lp} + R_{\hat\alpha\hat\beta}{}^{\pm}_{\hat\gamma}{}^{\hat\rho\hat\tau}M_{\hat\rho\hat\tau}\,. \tag{5.9c}$$

Here we have introduced the following definitions:

$$\begin{aligned}&R^{+}_{\hat\alpha}{}^{-}_{\hat\beta}{}^{\hat\gamma\hat\delta} := R^{i}_{\hat\alpha}{}^{j}_{\hat\beta}{}^{\hat\gamma\hat\delta}u^+_i u^-_j\,, \quad R^{+}_{\hat\alpha}{}^{-}_{\hat\beta}{}^{kl} := R^{i}_{\hat\alpha}{}^{j}_{\hat\beta}{}^{kl}u^+_i u^-_j\,, \\ &T_{\hat a}{}^{\pm}_{\hat\gamma}{}^{\hat\delta\pm} := T_{\hat a}{}^{k}_{\hat\gamma}{}^{\hat\delta l}u^\pm_k u^\pm_l\,, \quad R_{\hat a}{}^{\pm}_{\hat\gamma}{}^{\hat\rho\hat\tau} := R_{\hat a}{}^{k}_{\hat\gamma}{}^{\hat\rho\hat\tau}u^\pm_k\,, \quad R_{\hat a}{}^{\pm}_{\hat\gamma}{}^{lp} := R_{\hat a}{}^{k}_{\hat\gamma}{}^{lp}u^\pm_k\,,\end{aligned} \tag{5.10}$$

where the torsion and curvature tensors are given explicitly in section 2. In what follows, we often change the basis in the space of iso-tensors by the rule: $A^i \to A^\pm := A^i u^\pm_i$.

Let us return to the variation (5.7). We evaluate the anticommutators on the right of (5.7) with the aid of (5.9b). After that, all vector covariant derivative should be moved to the left by making use of (5.9c), and all SU(2)-generators should be moved to the right using (5.9a). If such transformations produce a spinor covariant derivative $\mathcal{D}^+_{\hat\alpha}$, it should be pushed to the right until it hits $\mathcal{L}^{++}$, and the latter vanishes due to the analyticity of the Lagrangian, $\mathcal{D}^+_{\hat\alpha}\mathcal{L}^{++}=0$. We end up with

$$
\begin{aligned}
\delta S_0 = -\frac{\varepsilon^{\hat\alpha\hat\beta\hat\gamma\hat\delta}}{96}\oint \mathrm{d}\mu^{++}\, b\int \mathrm{d}^5x\, e\Bigg[& 12\mathrm{i}(u^+u^-)\mathcal{D}_{\hat\alpha\hat\beta}\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\delta}+20(u^+u^-)T_{\hat\alpha\hat\beta}{}^-_{\hat\gamma}{}^{\hat\rho-}\mathcal{D}_{\hat\rho\hat\delta} \\
& +2\mathrm{i}R^+_{\hat\alpha}{}^-_{\hat\beta}{}^{--}\mathcal{D}_{\hat\gamma\hat\delta}+16\mathrm{i}T_{\hat\alpha\hat\beta}{}^-_{\hat\gamma}{}^{\hat\rho+}\mathcal{D}^-_{[\hat\rho}\mathcal{D}^-_{\hat\delta]}-\frac{4}{(u^+u^-)}R^+_{\hat\alpha}{}^-_{\hat\beta}{}^{+-}\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\delta} \\
& -4R^+_{\hat\alpha}{}^-_{\hat\beta}{}_{\hat\gamma}{}^{\hat\tau}\mathcal{D}^-_{[\hat\delta}\mathcal{D}^-_{\hat\tau]}+10\mathrm{i}R_{\hat\alpha\hat\beta}{}^-_{\hat\gamma}{}^{+-}\mathcal{D}^-_{\hat\delta}-10\mathrm{i}(u^+u^-)R_{\hat\alpha\hat\beta}{}^-_{\hat\gamma}{}_{\hat\delta}{}^{\hat\tau}\mathcal{D}^-_{\hat\tau} \\
& +6\mathrm{i}(\mathcal{D}^-_{\hat\alpha}T_{\hat\beta\hat\gamma}{}^-_{\hat\delta}{}^{\hat\rho+})\mathcal{D}^-_{\hat\rho}-\frac{2}{(u^+u^-)}(\mathcal{D}^-_{\hat\alpha}R^+_{\hat\beta}{}^-_{\hat\gamma}{}^{+-})\mathcal{D}^-_{\hat\delta}+2(\mathcal{D}^-_{\hat\alpha}R^+_{\hat\beta}{}^-_{\hat\gamma}{}_{\hat\delta}{}^{\hat\tau})\mathcal{D}^-_{\hat\tau} \\
+\Bigg(& 6R^+_{\hat\alpha}{}^-_{\hat\beta}{}^{kl}\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\delta}+8(\mathcal{D}^-_{\hat\alpha}R^+_{\hat\beta}{}^-_{\hat\gamma}{}^{kl})\mathcal{D}^-_{\hat\delta}-16\mathrm{i}(u^+u^-)R_{\hat\alpha\hat\beta}{}^-_{\hat\gamma}{}^{kl}\mathcal{D}^-_{\hat\delta}+3(\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta}R^+_{\hat\gamma}{}^-_{\hat\delta}{}^{kl}) \\
& -6\mathrm{i}(u^+u^-)(\mathcal{D}^-_{\hat\alpha}R_{\hat\beta\hat\gamma}{}^-_{\hat\delta}{}^{kl})-10\mathrm{i}T_{\hat\alpha\hat\beta}{}^-_{\hat\gamma}{}^{\hat\rho-}R^+_{\hat\rho}{}^-_{\hat\delta}{}^{kl}+2\mathrm{i}T_{\hat\alpha\hat\beta}{}^-_{\hat\gamma}{}^{\hat\rho+}R^-_{\hat\rho}{}^-_{\hat\delta}{}^{kl} \\
& +\frac{1}{(u^+u^-)}R^+_{\hat\alpha}{}^-_{\hat\beta}{}^{--}R^+_{\hat\gamma}{}^-_{\hat\delta}{}^{kl}-R^+_{\hat\alpha}{}^-_{\hat\beta}{}_{\hat\gamma}{}^{\hat\rho}R^-_{\hat\rho}{}^-_{\hat\delta}{}^{kl}\Bigg)J_{kl}\Bigg]\mathcal{L}^{++}\ . \qquad (5.11)
\end{aligned}
$$

Let us analyze the contributions to the right-hand side of (5.11), which are proportional to the SU(2)-generators J_{kl}. It is important to note that all the coefficients in front of J_{kl} are homogeneous functions of degree 1 in the variables u^+_i, and of degree 3 in u^-_i. This follows from the fact that such terms come from the variation $\delta(\mathcal{D}^-)^4$ which results in replacing one of the four isotwistors (u^-)'s by $(b\,u^+)$. Another piece of useful information is the fact that the lagrangian $\mathcal{L}^{++}$ is a projective superfield of weight 2, and hence

$$J_{kl}\mathcal{L}^{++} = -\frac{1}{(u^+u^-)}\Big(u^+_{(k}u^+_{l)}D^{--}-2u^+_{(k}u^-_{l)}\Big)\mathcal{L}^{++}\,, \qquad (5.12a)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^{++} = 2\frac{(\dot{u}^+u^-)}{(u^+u^-)}\mathcal{L}^{++}-\frac{(\dot{u}^+u^+)}{(u^+u^-)}D^{--}\mathcal{L}^{++}\,, \qquad (5.12b)$$

$$(\dot{u}^+u^+)J_{kl}\mathcal{L}^{++} = u^+_{(k}u^+_{l)}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^{++}-2\frac{(\dot{u}^+u^-)}{(u^+u^-)}u^+_{(k}u^+_{l)}\mathcal{L}^{++}+2\frac{(\dot{u}^+u^+)}{(u^+u^-)}u^+_{(k}u^-_{l)}\mathcal{L}^{++}\ . \qquad (5.12c)$$

The latter result leads to

$$\frac{(\dot{u}^+u^+)}{(u^+u^-)^4}\,b\,J_{kl}\mathcal{L}^{++} = \frac{\mathrm{d}}{\mathrm{d}t}\left[b\frac{u^+_{(k}u^+_{l)}}{(u^+u^-)^4}\mathcal{L}^{++}\right]+4b\frac{(\dot{u}^+u^+)}{(u^+u^-)^5}u^+_{(k}u^-_{l)}\mathcal{L}^{++}+b\frac{(\dot{u}^+u^-)}{(u^+u^-)^5}u^+_{(k}u^+_{l)}\mathcal{L}^{++}\ . \qquad (5.13)$$

This implies that, given an operator $\mathcal{O}^{(kl)}=\mathcal{O}^{(lk)}$ which is an homogenous function of degree 1 in u^+_i (as in our case), the following equation holds

$$\oint \mathrm{d}\mu^{++}\, b\,\mathcal{O}^{(kl)}J_{kl}\mathcal{L}^{++} = \oint \mathrm{d}\mu^{++}\left\{\frac{4b\,\mathcal{O}^{+-}}{(u^+u^-)}\mathcal{L}^{++}+\frac{b\,u^+_k u^+_l}{(u^+u^-)}\Big(D^{--}\mathcal{O}^{(kl)}\Big)\mathcal{L}^{++}\right\}\ . \qquad (5.14)$$

Now, it remains to make use of the explicit expressions for the torsion and curvature tensors, see eqs. (2.12a), (2.12c), as well as to notice the relations

$$\mathcal{D}^-_{\hat\alpha}S^{+-} = -\frac{(u^+u^-)}{4}\Big(3\mathcal{F}^-_{\hat\alpha}+\mathcal{N}^-_{\hat\alpha}\Big)\,,\quad \mathcal{D}^{\hat\alpha-}F_{\hat\alpha}{}^{\hat\beta}=\frac{5}{2}\mathcal{F}^{\hat\beta-}\,,\quad \mathcal{D}^{\hat\alpha-}N_{\hat\alpha}{}^{\hat\beta}=\frac{5}{2}\mathcal{N}^{\hat\beta-}\,, \qquad (5.15)$$

After some computations, one obtains

$$\delta S_0 = \oint \mathrm{d}\mu^{++}\, b \int \mathrm{d}^5x\, e \left[\frac{\mathrm{i}(u^+u^-)}{4}\mathcal{D}^{\hat\alpha\hat\beta}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta} + \frac{25\mathrm{i}}{12}S^{+-}(\mathcal{D}^-)^2 \right.$$
$$\left. - 5\mathrm{i}(u^+u^-)\mathcal{F}^{\hat\alpha -}\mathcal{D}^-_{\hat\alpha} - 22\, S^{--}S^{+-} \right]\mathcal{L}^{++}\Big| \,. \qquad (5.16)$$

An important remark is in order. The original variation δS_0 contained numerous contributions proportional to $(u^+u^-)\mathcal{N}^{\hat\alpha -}\mathcal{D}^-_{\hat\alpha}$. All such terms have cancelled out. Although at first sight such non-trivial cancellations may appear miraculous, there is a simple explanation for that. The point is that such contributions to the projective variation of S are impossible to cancel by means of adding some "counterterms" to the action. Complete cancellation is the only option compatible with projective invariance.

To cancel out the second and fourth terms in (5.16), we add to S_0 the following functional:

$$S_1 = \oint \mathrm{d}\mu^{++} \int \mathrm{d}^5x\, e \Big[-\frac{25\mathrm{i}}{24}S^{--}(\mathcal{D}^-)^2 + 18\, S^{--}S^{--} \Big]\mathcal{L}^{++}\Big| \,, \qquad (5.17)$$

Evaluating the projective variation of $S_0 + S_1$ gives

$$\delta\Big(S_0 + S_1\Big) = \oint \mathrm{d}\mu^{++}\, b \int \mathrm{d}^5x\, e \left[\frac{\mathrm{i}(u^+u^-)}{4}\mathcal{D}^{\hat\alpha\hat\beta}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta} - 5\mathrm{i}(u^+u^-)\mathcal{F}^{\hat\alpha -}\mathcal{D}^-_{\hat\alpha} \right]\mathcal{L}^{++}\Big| \,. \qquad (5.18)$$

To simplify the last variation, we have to start using the relations that hold in WZ gauge of section 4. In particular, making use of (4.1) gives

$$\delta\Big(S_0 + S_1\Big) = \oint \mathrm{d}\mu^{++}\, b \int \mathrm{d}^5x\, e \left[\frac{\mathrm{i}(u^+u^-)}{4}\nabla^{\hat\alpha\hat\beta}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta} + \frac{\mathrm{i}}{4}\Psi^{\hat\alpha\hat\beta\hat\gamma -}[\mathcal{D}^+_{\hat\gamma}, \mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta}] \right.$$
$$\left. -\frac{\mathrm{i}}{4}\Psi^{\hat\alpha\hat\beta\hat\gamma +}\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta} + \frac{\mathrm{i}}{4}\phi^{\hat\alpha\hat\beta --}\{\mathcal{D}^+_{\hat\alpha}, \mathcal{D}^-_{\hat\beta}\} + \frac{\mathrm{i}}{2}\phi^{\hat\alpha\hat\beta +-}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta} - 5\mathrm{i}(u^+u^-)\mathcal{F}^{\hat\alpha -}\mathcal{D}^-_{\hat\alpha} \right]\mathcal{L}^{++}\Big| . \qquad (5.19)$$

Here, the operators proportional to the connection ϕ can be seen to cancel out by adding the functional

$$S_2 = \oint \mathrm{d}\mu^{++} \int \mathrm{d}^5x\, e \left[-\frac{\mathrm{i}}{4}\phi^{\hat\alpha\hat\beta --}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta} \right]\mathcal{L}^{++}\Big| \,. \qquad (5.20)$$

Futhermore, in order to cancel out the first term in the second line of (5.19), it is necessary to add one more "counterterm":

$$S_3 = \oint \mathrm{d}\mu^{++} \int \mathrm{d}^5x\, e \left[\frac{\mathrm{i}}{4}\Psi^{\hat\alpha\hat\beta\hat\gamma -}\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta} \right]\mathcal{L}^{++}\Big| \,. \qquad (5.21)$$

Now, the projective variation of $S_0 + S_1 + S_2 + S_3$ is

$$\delta\Big(S_0 + S_1 + S_2 + S_3\Big) = \oint \mathrm{d}\mu^{++}\, b \int \mathrm{d}^5x\, e \left[\frac{\mathrm{i}(u^+u^-)}{4}\nabla^{\hat\alpha\hat\beta}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta} \right.$$
$$\left. + \frac{\mathrm{i}}{2}\Psi^{\hat\alpha\hat\beta\hat\gamma -}[\mathcal{D}^+_{\hat\gamma}, \mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta}] + \frac{\mathrm{i}}{4}\Psi^{\hat\alpha\hat\beta\hat\gamma -}\mathcal{D}^-_{\hat\gamma}\{\mathcal{D}^+_{\hat\alpha}, \mathcal{D}^-_{\hat\beta}\} - 5\mathrm{i}(u^+u^-)\mathcal{F}^{\hat\alpha -}\mathcal{D}^-_{\hat\alpha} \right]\mathcal{L}^{++}\Big| \,. \qquad (5.22)$$

To simplify the variation obtained, we compute the (anti)commutators in (5.22). In this way, we will produce terms with vector covariant derivatives, $\mathcal{D}_{\hat{a}}$, and also terms with the Lorentz and SU(2)-generators. Then we should systematically move all the covariant derivatives $\mathcal{D}_{\hat{a}}$ to the left, with the aid of the algebra of covariant derivatives, and finally make use of the WZ gauge relation (4.1). Similarly, we should systematically move all the generators to the right using (5.14) and $M_{\hat{\alpha}\hat{\beta}}\mathcal{L}^{++} = 0$. As a result, eq. (5.22) turns into

$$
\begin{aligned}
\delta\Big(S_0+S_1+S_2+S_3\Big) = \oint \mathrm{d}\mu^{++}\, b \int \mathrm{d}^5x\, e \Bigg[& \frac{\mathrm{i}(u^+u^-)}{4}(\Gamma^{\hat{a}})^{\hat{\gamma}\hat{\delta}}\nabla_{\hat{a}}\mathcal{D}^-_{\hat{\gamma}}\mathcal{D}^-_{\hat{\delta}} \\
& -4(u^+u^-)(\Sigma^{\hat{a}\hat{b}})_{\hat{\beta}}{}^{\hat{\gamma}}\nabla_{[\hat{a}}\Psi_{\hat{b}]}{}^{\hat{\beta}-}\mathcal{D}^-_{\hat{\gamma}} + 4(u^+u^-)(\Sigma^{\hat{a}\hat{b}})_{\hat{\beta}}{}^{\hat{\gamma}}(\nabla_{[\hat{a}}\Psi_{\hat{b}]}{}^{\hat{\beta}-})\mathcal{D}^-_{\hat{\gamma}} - 12(\Sigma^{\hat{a}\hat{b}})^{\hat{\alpha}}{}_{\hat{\gamma}}\phi_{[\hat{a}}{}^{+-}\Psi_{\hat{b}]}{}^{\hat{\gamma}-}\mathcal{D}^-_{\hat{\alpha}} \\
& -4(\Sigma^{\hat{a}\hat{b}})_{\hat{\gamma}}{}^{\hat{\beta}}\Psi_{\hat{a}}{}^{\hat{\gamma}-}\Psi_{\hat{b}}{}^{\hat{\delta}+}\mathcal{D}^-_{[\hat{\delta}}\mathcal{D}^-_{\hat{\beta}]} + 4(\Sigma^{\hat{a}\hat{b}})_{\hat{\gamma}}{}^{\hat{\beta}}\Psi_{\hat{a}}{}^{\hat{\gamma}-}\Psi_{\hat{b}}{}^{\hat{\delta}-}\{\mathcal{D}^+_{\hat{\delta}},\mathcal{D}^-_{\hat{\beta}}\} - 2(\Sigma^{\hat{a}\hat{b}})_{\hat{\gamma}}{}^{\hat{\beta}}\Psi_{\hat{a}}{}^{\hat{\gamma}-}\Psi_{\hat{b}}{}^{\hat{\delta}+}\{\mathcal{D}^-_{\hat{\delta}},\mathcal{D}^-_{\hat{\beta}}\} \\
& + \Psi^{\hat{\alpha}\hat{\beta}\hat{\gamma}-}\Bigg(\frac{7\mathrm{i}}{2(u^+u^-)}R^+_{\hat{\gamma}}{}^-_{\hat{\alpha}}{}^{+-}\mathcal{D}^-_{\hat{\beta}} + \frac{\mathrm{i}}{(u^+u^-)}R^+_{\hat{\alpha}}{}^-_{\hat{\beta}}{}^{+-}\mathcal{D}^-_{\hat{\gamma}} + \frac{\mathrm{i}}{(u^+u^-)}R^-_{\hat{\gamma}}{}^-_{\hat{\alpha}}{}^{++}\mathcal{D}^-_{\hat{\beta}} \\
& \qquad + \frac{\mathrm{i}}{4(u^+u^-)}R^-_{\hat{\alpha}}{}^-_{\hat{\beta}}{}^{++}\mathcal{D}^-_{\hat{\gamma}} + \frac{\mathrm{i}}{2}R^+_{\hat{\gamma}}{}^-_{\hat{\alpha}}{}_{\hat{\beta}}{}^{\hat{\rho}}\mathcal{D}^-_{\hat{\rho}} + T_{\hat{\gamma}\hat{\beta}}{}^-_{\hat{\alpha}}{}^{\hat{\delta}+}\mathcal{D}^-_{\hat{\delta}} - \frac{1}{2}T_{\hat{\alpha}\hat{\beta}}{}^-_{\hat{\gamma}}{}^{\hat{\delta}+}\mathcal{D}^-_{\hat{\delta}} \\
& \qquad - \frac{2\mathrm{i}}{(u^+u^-)}(\mathcal{D}^-_{\hat{\alpha}}R^+_{\hat{\gamma}}{}^-_{\hat{\beta}}{}^{+-}) - \frac{\mathrm{i}}{2(u^+u^-)}(\mathcal{D}^-_{\hat{\alpha}}R^-_{\hat{\gamma}}{}^-_{\hat{\beta}}{}^{++}) + \frac{\mathrm{i}}{(u^+u^-)}(\mathcal{D}^-_{\hat{\gamma}}R^+_{\hat{\alpha}}{}^-_{\hat{\beta}}{}^{+-}) \\
& \qquad + \frac{\mathrm{i}}{4(u^+u^-)}(\mathcal{D}^-_{\hat{\gamma}}R^-_{\hat{\alpha}}{}^-_{\hat{\beta}}{}^{++}) - 4R_{\hat{\gamma}\hat{\beta}}{}^-_{\hat{\alpha}}{}^{+-} + 2R_{\hat{\alpha}\hat{\beta}}{}^-_{\hat{\gamma}}{}^{+-}\Bigg) \\
& \qquad\qquad -5\mathrm{i}(u^+u^-)\mathcal{F}^{\hat{\alpha}-}\mathcal{D}^-_{\hat{\alpha}}\Bigg]\mathcal{L}^{++}\Big| \,.
\end{aligned} \tag{5.23}
$$

In the variation obtained, the first three terms can be simplified by using some relations that hold in the WZ gauge. In particular, the first two terms in (5.23) are of the form $\nabla_{\hat{a}}U^{\hat{a}}$, for some $U^{\hat{a}}$, and can be simplified by using the rule for integration by parts (4.7). Furthermore, the third term in (5.23) can be transformed to the form:

$$
\begin{aligned}
(\Sigma^{\hat{a}\hat{b}})^{\hat{\alpha}}{}_{\hat{\gamma}}\big(\nabla_{[\hat{a}}\Psi_{\hat{b}]}{}^{\hat{\gamma}-}\big) = {} & \frac{5\mathrm{i}}{4}\mathcal{F}^{\hat{\alpha}-} - \frac{1}{(u^+u^-)}(\Sigma^{\hat{a}\hat{b}})^{\hat{\alpha}}{}_{\hat{\gamma}}\phi_{[\hat{a}}{}^{--}\Psi_{\hat{b}]}{}^{\hat{\gamma}+} + \frac{1}{(u^+u^-)}(\Sigma^{\hat{a}\hat{b}})^{\hat{\alpha}}{}_{\hat{\gamma}}\phi_{[\hat{a}}{}^{+-}\Psi_{\hat{b}]}{}^{\hat{\gamma}-} \\
& + \frac{1}{2}(\Sigma^{\hat{a}\hat{b}})^{\hat{\alpha}}{}_{\hat{\gamma}}\mathcal{T}_{\hat{a}\hat{b}}{}^{\hat{c}}\Psi_{\hat{c}}{}^{\hat{\gamma}-} - \frac{1}{(u^+u^-)}(\Sigma^{\hat{a}\hat{b}})^{\hat{\alpha}}{}_{\hat{\gamma}}\Psi_{[\hat{a}}{}^{\hat{\beta}+}T_{\hat{b}]}{}^-_{\hat{\beta}}{}^{\hat{\gamma}-} \\
& + \frac{1}{(u^+u^-)}(\Sigma^{\hat{a}\hat{b}})^{\hat{\alpha}}{}_{\hat{\gamma}}\Psi_{[\hat{a}}{}^{\hat{\beta}-}T_{\hat{b}]}{}^+_{\hat{\beta}}{}^{\hat{\gamma}-}\,,
\end{aligned} \tag{5.24}
$$

as a consequence of the identity (4.5b). Here the space-time torsion is given by eq. (4.5a) which can be equivalently rewritten as follows:

$$
\mathcal{T}_{\hat{a}\hat{b}}{}^{\hat{c}} = \frac{4\mathrm{i}}{(u^+u^-)}(\Gamma^{\hat{c}})_{\hat{\gamma}\hat{\delta}}\Psi_{[\hat{a}}{}^{\hat{\gamma}-}\Psi_{\hat{b}]}{}^{\hat{\delta}+}\,. \tag{5.25}
$$

Further calculations lead to

$$
\begin{aligned}
&\delta\Big(S_0+S_1+S_2+S_3\Big)= \\
&=\oint \mathrm{d}\mu^{++}\, b\int \mathrm{d}^5x\, e\bigg[\,2(\Sigma^{\hat a\hat b})_{\hat\beta}{}^{\hat\gamma}\Psi_{[\hat a}{}^{\hat\beta+}\Psi_{\hat b]}{}^{\hat\delta-}\mathcal{D}^-_{[\hat\gamma}\mathcal{D}^-_{\hat\delta]}+2(\Sigma^{\hat a\hat b})_{\hat\beta}{}^{\hat\gamma}\Psi_{[\hat a}{}^{\hat\beta-}\Psi_{\hat b]}{}^{\hat\delta+}\mathcal{D}^-_{[\hat\gamma}\mathcal{D}^-_{\hat\delta]} \\
&\quad+4(\Sigma^{\hat a\hat b})_{\hat\beta}{}^{\hat\gamma}\Psi_{[\hat a}{}^{\hat\beta-}\Psi_{\hat b]}{}^{\hat\delta-}\{\mathcal{D}^+_{\hat\delta},\mathcal{D}^-_{\hat\gamma}\}-2(\Sigma^{\hat a\hat b})_{\hat\beta}{}^{\hat\gamma}\Psi_{[\hat a}{}^{\hat\beta-}\Psi_{\hat b]}{}^{\hat\delta+}\{\mathcal{D}^-_{\hat\delta},\mathcal{D}^-_{\hat\gamma}\} \\
&\quad+4\Psi^{\hat\alpha\hat\beta+}_{\hat\beta}S^{--}\mathcal{D}^-_{\hat\alpha}+8\Psi^{\hat\alpha\hat\beta-}_{\hat\beta}S^{+-}\mathcal{D}^-_{\hat\alpha}-4(\Sigma^{\hat a\hat b})^{\hat\alpha}{}_{\hat\gamma}\phi_{[\hat a}{}^{--}\Psi_{\hat b]}{}^{\hat\gamma+}\mathcal{D}^-_{\hat\alpha}-8(\Sigma^{\hat a\hat b})^{\hat\alpha}{}_{\hat\gamma}\phi_{[\hat a}{}^{+-}\Psi_{\hat b]}{}^{\hat\gamma-}\mathcal{D}^-_{\hat\alpha} \\
&\quad+16\mathrm{i}(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}(\Gamma^{\hat c})_{\hat\gamma\hat\delta}\Psi_{[\hat a}{}^{\hat\gamma-}\Psi_{\hat c]}{}^{\hat\delta+}\Psi_{\hat b}{}^{\hat\alpha-}\mathcal{D}^{\hat\beta-}-8\mathrm{i}(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}(\Gamma^{\hat c})_{\hat\gamma\hat\delta}\Psi_{[\hat a}{}^{\hat\gamma-}\Psi_{\hat b]}{}^{\hat\delta+}\Psi_{\hat c}{}^{\hat\alpha-}\mathcal{D}^{\hat\beta-} \\
&\quad+9(u^+u^-)(\Gamma^{\hat a})^{\hat\beta\hat\gamma}\Psi_{\hat a}{}^{\hat\alpha-}\Xi_{\hat\beta\hat\alpha\hat\gamma}{}^{-}-18(u^+u^-)(\Gamma^{\hat a})_{\hat\alpha}{}^{\hat\gamma}\Psi_{\hat a}{}^{\hat\alpha-}\mathcal{F}_{\hat\gamma}{}^{-}\bigg]\mathcal{L}^{++}\bigg| \,.
\end{aligned}
\tag{5.26}
$$

Note that the term $-5\mathrm{i}(u^+u^-)\mathcal{F}^{\hat\alpha-}\mathcal{D}^-_{\hat\alpha}$ in (5.23), which cannot be consistently produced by the variation of any Lagrangian, has cancelled out at this point.

The terms in the fourth line of (5.26) can be seen to cancel out by adding to the action the following functional:

$$
S_4=\oint \mathrm{d}\mu^{++}\int \mathrm{d}^5x\, e\bigg[\,4(\Sigma^{\hat a\hat b})^{\hat\alpha}{}_{\hat\gamma}\phi_{[\hat a}{}^{--}\Psi_{\hat b]}{}^{\hat\gamma-}\mathcal{D}^-_{\hat\alpha}-4\Psi^{\hat\alpha\hat\beta-}_{\hat\beta}S^{--}\mathcal{D}^-_{\hat\alpha}\bigg]\mathcal{L}^{++}\bigg| \,. \tag{5.27}
$$

In addition, in order to cancel out the two terms quadratic in spinor derivatives $\mathcal{D}^-$ in the second line of (5.26), one has to add to S one more "counterterm"

$$
S_5=\oint \mathrm{d}\mu^{++}\int \mathrm{d}^5x\, e\bigg[-2(\Sigma^{\hat a\hat b})_{\hat\beta}{}^{\hat\gamma}\Psi_{\hat a}{}^{\hat\beta-}\Psi_{\hat b}{}^{\hat\delta-}\mathcal{D}^-_{[\hat\gamma}\mathcal{D}^-_{\hat\delta]}\bigg]\mathcal{L}^{++}\bigg| \,. \tag{5.28}
$$

At this point, we can simplify $\delta\Big(S_0+S_1+S_2+S_3+S_4+S_5\Big)$ by computing the remaining anticommutators and then using the same strategy as before, that is: (i) systematically move all the covariant derivatives $\mathcal{D}_{\hat a}$ to the left , using the algebra of covariant derivatives, and then we apply the relation (4.1); (ii) systematically move to the right all the generators using (5.14) and $M_{\hat\alpha\hat\beta}\mathcal{L}^{++}=0$. Such calculations give

$$
\begin{aligned}
&\delta\Big(S_0+S_1+S_2+S_3+S_4+S_5\Big)= \\
&=\oint \mathrm{d}\mu^{++}\, b\int \mathrm{d}^5x\, e\bigg[-24\mathrm{i}(u^+u^-)(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\gamma}\Psi_{[\hat a}{}^{(\hat\alpha-}\Psi_{\hat b]}{}^{\hat\beta)-}\Big(F_{\hat\beta}{}^{\hat\gamma}+N_{\hat\beta}{}^{\hat\gamma}+\frac{3}{(u^+u^-)}\delta^{\hat\gamma}_{\hat\beta}S^{+-}\Big) \\
&\quad+6\mathrm{i}(u^+u^-)\varepsilon^{\hat a\hat b\hat c\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}\nabla_{\hat c}\Psi_{[\hat a}{}^{(\hat\alpha-}\Psi_{\hat b]}{}^{\hat\beta)-}+12\mathrm{i}(u^+u^-)\varepsilon^{\hat a\hat b\hat c\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}\Psi_{\hat a}{}^{\hat\alpha-}(\nabla_{[\hat b}\Psi_{\hat c]}{}^{\hat\beta-}) \\
&\quad+24\mathrm{i}\varepsilon^{\hat a\hat b\hat c\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}\Psi_{[\hat a}{}^{(\hat\alpha-}\Psi_{\hat b]}{}^{\hat\beta)-}\phi_{\hat c}{}^{+-}-6\mathrm{i}\varepsilon^{\hat a\hat b\hat c\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}\Psi_{[\hat a}{}^{(\hat\alpha-}\Psi_{\hat b]}{}^{\hat\beta)-}\Psi_{\hat c}{}^{\hat\gamma+}\mathcal{D}^-_{\hat\gamma} \\
&\quad+16\mathrm{i}(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}(\Gamma^{\hat c})_{\hat\gamma\hat\delta}\Psi_{[\hat a}{}^{\hat\gamma-}\Psi_{\hat c]}{}^{\hat\delta+}\Psi_{\hat b}{}^{\hat\alpha-}\mathcal{D}^{\hat\beta-}-8\mathrm{i}(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}(\Gamma^{\hat c})_{\hat\gamma\hat\delta}\Psi_{[\hat a}{}^{\hat\gamma-}\Psi_{\hat b]}{}^{\hat\delta+}\Psi_{\hat c}{}^{\hat\alpha-}\mathcal{D}^{\hat\beta-} \\
&\quad+9(u^+u^-)(\Gamma^{\hat a})^{\hat\beta\hat\gamma}\Psi_{\hat a}{}^{\hat\alpha-}\Xi_{\hat\beta\hat\alpha\hat\gamma}{}^{-}-18(u^+u^-)(\Gamma^{\hat a})_{\hat\alpha}{}^{\hat\gamma}\Psi_{\hat a}{}^{\hat\alpha-}\mathcal{F}_{\hat\gamma}{}^{-}\bigg]\mathcal{L}^{++}\bigg| \,.
\end{aligned}
\tag{5.29}
$$

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Here the third line can be simplified by using the integration by parts (4.7) and the equation

$$
\begin{aligned}
\nabla_{[\hat a}\Psi_{\hat b]}{}^{\hat\gamma-} = {}& \frac{\mathrm{i}}{4}\mathcal{D}^{\hat\gamma-}F_{\hat a\hat b} + \frac{2\mathrm{i}}{(u^+u^-)}(\Gamma^{\hat c})_{\hat\gamma\hat\delta}\Psi_{[\hat a}{}^{\hat\gamma-}\Psi_{\hat b]}{}^{\hat\delta+}\Psi_{\hat c}{}^{\hat\gamma-} + \frac{1}{(u^+u^-)}\phi_{[\hat a}{}^{+-}\Psi_{\hat b]}{}^{\hat\gamma-} \\
& -\frac{1}{(u^+u^-)}\phi_{[\hat a}{}^{--}\Psi_{\hat b]}{}^{\hat\gamma+} - \frac{1}{(u^+u^-)}\Psi_{[\hat a}{}^{\hat\beta+}T_{\hat b]}{}_{\hat\beta}^{-\hat\gamma-} + \frac{1}{(u^+u^-)}\Psi_{[\hat a}{}^{\hat\beta-}T_{\hat b]}{}_{\hat\beta}^{+\hat\gamma-}\,, \qquad (5.30)
\end{aligned}
$$

which follows from (4.5b).

After some computations, the variation becomes

$$
\begin{aligned}
&\delta\Big(S_0+S_1+S_2+S_3+S_4+S_5\Big) = \\
&= \oint \mathrm{d}\mu^{++}\, b \int \mathrm{d}^5x\, e \Bigg[-36\mathrm{i}(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}\Big(\Psi_{\hat a}{}^{\hat\alpha-}\Psi_{\hat b}{}^{\hat\beta-}S^{+-} + \Psi_{\hat a}{}^{\hat\alpha+}\Psi_{\hat b}{}^{\hat\beta-}S^{--}\Big) \\
&\quad +12\mathrm{i}\varepsilon^{\hat a\hat b\hat c\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}\Big(\Psi_{\hat a}{}^{\hat\alpha-}\Psi_{\hat b}{}^{\hat\beta-}\phi_{\hat c}{}^{+-} + \Psi_{\hat a}{}^{\hat\alpha-}\Psi_{\hat b}{}^{\hat\beta+}\phi_{\hat c}{}^{--}\Big) \\
&\quad +24\varepsilon^{\hat a\hat b\hat c\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}(\Gamma^{\hat d})_{\hat\gamma\hat\delta}\Big(\Psi_{\hat a}{}^{\hat\alpha-}\Psi_{\hat b}{}^{\hat\beta-}\Psi_{[\hat d}{}^{\hat\gamma-}\Psi_{\hat c]}{}^{\hat\delta+} + \Psi_{\hat a}{}^{\hat\alpha-}\Psi_{\hat b}{}^{\hat\gamma-}\Psi_{\hat d}{}^{\hat\beta-}\Psi_{\hat c}{}^{\hat\delta+}\Big) \\
&\quad -6\mathrm{i}\varepsilon^{\hat a\hat b\hat c\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}\Psi_{[\hat a}{}^{(\hat\alpha-}\Psi_{\hat b]}{}^{\hat\beta)-}\Psi_{\hat c}{}^{\hat\gamma+}\mathcal{D}^-_{\hat\gamma} + 16\mathrm{i}(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}(\Gamma^{\hat c})_{\hat\gamma\hat\delta}\Psi_{[\hat a}{}^{\hat\gamma-}\Psi_{\hat c]}{}^{\hat\delta+}\Psi_{\hat b}{}^{\hat\alpha-}\mathcal{D}^{\hat\beta-} \\
&\quad -8\mathrm{i}(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}(\Gamma^{\hat c})_{\hat\gamma\hat\delta}\Psi_{[\hat a}{}^{\hat\gamma-}\Psi_{\hat b]}{}^{\hat\delta+}\Psi_{\hat c}{}^{\hat\alpha-}\mathcal{D}^{\hat\beta-}\Bigg]\mathcal{L}^{++}\Big| \,. \qquad (5.31)
\end{aligned}
$$

To cancel out the expressions in the second and third lines of (5.31), we have to add to the action the following functional:

$$
S_6 = \oint \mathrm{d}\mu^{++}\int \mathrm{d}^5x\, e\Bigg[\Psi_{\hat a}{}^{\hat\alpha-}\Psi_{\hat b}{}^{\hat\beta-}\Big(18\mathrm{i}(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}S^{--} - 6\mathrm{i}\varepsilon^{\hat a\hat b\hat c\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}\phi_{\hat c}{}^{--}\Big)\Bigg]\mathcal{L}^{++}\Big| \,. \qquad (5.32)
$$

Now, let us turn our attention to the three gravitini in (5.31). For their analysis, we need two auxiliary resuts. First, for any tensor $A_{\hat a\hat b\hat c} = -A_{\hat b\hat a\hat c}$, it holds

$$
(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}A_{\hat a\hat c\hat b} = -\frac{3}{2}(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}A_{[\hat a\hat b\hat c]} + \frac{1}{2}(\Sigma^{\hat a\hat b})_{\hat\alpha\hat\beta}A_{\hat a\hat b\hat c}\,, \qquad (5.33)
$$

Given an antisymmetric tensor, $A^{\hat d\hat e} = -A^{\hat e\hat d}$, it holds

$$
\varepsilon_{\hat a\hat b\hat c\hat d\hat e}(\Gamma^{\hat a})_{\hat\alpha\hat\beta}(\Sigma^{\hat b\hat c})_{\hat\gamma\hat\delta}A^{\hat d\hat e} = 4\varepsilon_{\hat\alpha\hat\beta}A_{\hat\gamma\hat\delta} - 4\varepsilon_{\hat\alpha\hat\gamma}A_{\hat\beta\hat\delta} - 4\varepsilon_{\hat\alpha\hat\delta}A_{\hat\beta\hat\gamma} + 4\varepsilon_{\hat\beta\hat\gamma}A_{\hat\alpha\hat\delta} + 4\varepsilon_{\hat\beta\hat\delta}A_{\hat\alpha\hat\gamma}\,. \qquad (5.34)
$$

With the aid of these identities, the contributions proportional to three gravitini in (5.31) can be seen to be equivalent to

$$
-2\mathrm{i}\oint \mathrm{d}\mu^{++}\, b\int \mathrm{d}^5x\, e\,\varepsilon^{\hat a\hat b\hat c\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}\Big(\Psi_{\hat a}{}^{\hat\alpha-}\Psi_{\hat b}{}^{\hat\beta-}\Psi_{\hat c}{}^{\hat\gamma+} + 2\Psi_{\hat a}{}^{\hat\alpha+}\Psi_{\hat b}{}^{\hat\beta-}\Psi_{\hat c}{}^{\hat\gamma-}\Big)\mathcal{D}^-_{\hat\gamma}\mathcal{L}^{++}\Big| \,. \qquad (5.35)
$$

These terms identically are cancelled out against the projective variation of the functional

$$
S_7 = 2\mathrm{i}\oint \mathrm{d}\mu^{++}\int \mathrm{d}^5x\, e\,\varepsilon^{\hat a\hat b\hat c\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}\Psi_{\hat a}{}^{\hat\alpha-}\Psi_{\hat b}{}^{\hat\beta-}\Psi_{\hat c}{}^{\hat\gamma-}\mathcal{D}^-_{\hat\gamma}\,\mathcal{L}^{++}\Big| \,. \qquad (5.36)
$$

Finally, it can be seen that the terms with four gravitini in (5.31) cancel each other. As a result, we obtain

$$\delta\Big(S_0 + S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7\Big) = 0 \ . \tag{5.37}$$

The action (5.3) has been proved to be projective invariant. There is no need to demonstrate its invariance under the local supersymmetry transformations, since (5.3) is simply the component form of the locally supersymmetric action (3.13) in the WZ gauge.

Various supergravity-matter systems correspond to different choices for $\mathcal{L}^{++}$. Explicit examples of such dynamical systems are given in [1].

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A. 5D conventions

Our 5D notations and conventions correspond to [26]. The Minkowski metric is given by $\eta_{\hat{m}\hat{n}} = \mathrm{diag}\{-1,1,1,1,1\}$ $(\hat{m},\hat{n} = 0,1,2,3,5)$. The 5D gamma-matrices $\Gamma_{\hat{m}} = (\Gamma_m, \Gamma_5)$, with $m = 0,1,2,3$, are defined by

$$\{\Gamma_{\hat{m}}\, , \, \Gamma_{\hat{n}}\} = -2\eta_{\hat{m}\hat{n}}\, \mathbb{1}\, , \qquad (\Gamma_{\hat{m}})^\dagger = \Gamma_0\, \Gamma_{\hat{m}}\, \Gamma_0 \tag{A.1}$$

are chosen in accordance with

$$(\Gamma_m)_{\hat{\alpha}}{}^{\hat{\beta}} = \begin{pmatrix} 0 & (\sigma_m)_{\alpha\dot{\beta}} \\ (\tilde{\sigma}_m)^{\dot{\alpha}\beta} & 0 \end{pmatrix}, \qquad (\Gamma_5)_{\hat{\alpha}}{}^{\hat{\beta}} = \begin{pmatrix} -\mathrm{i}\,\delta_\alpha{}^\beta & 0 \\ 0 & \mathrm{i}\,\delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix}, \tag{A.2}$$

such that $\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_5 = \mathbb{1}$. The charge conjugation matrix, $C = (\varepsilon^{\hat{\alpha}\hat{\beta}})$, and its inverse, $C^{-1} = C^\dagger = (\varepsilon_{\hat{\alpha}\hat{\beta}})$ are defined by

$$C\,\Gamma_{\hat{m}}\, C^{-1} = (\Gamma_{\hat{m}})^{\mathrm{T}}\, , \qquad \varepsilon^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & -\varepsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \quad \varepsilon_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & -\varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \tag{A.3}$$

The antisymmetric matrices $\varepsilon^{\hat{\alpha}\hat{\beta}}$ and $\varepsilon_{\hat{\alpha}\hat{\beta}}$ are used to raise and lower the four-component spinor indices.

A Dirac spinor, $\Psi = (\Psi_{\hat{\alpha}})$, and its Dirac conjugate, $\bar{\Psi} = (\bar{\Psi}^{\hat{\alpha}}) = \Psi^\dagger\, \Gamma_0$, look like

$$\Psi_{\hat{\alpha}} = \begin{pmatrix} \psi_\alpha \\ \bar{\phi}^{\dot{\alpha}} \end{pmatrix}, \qquad \bar{\Psi}^{\hat{\alpha}} = (\phi^\alpha\, , \, \bar{\psi}_{\dot{\alpha}})\ . \tag{A.4}$$

One can now combine $\bar{\Psi}^{\hat{\alpha}} = (\phi^\alpha, \bar{\psi}_{\dot{\alpha}})$ and $\Psi^{\hat{\alpha}} = \varepsilon^{\hat{\alpha}\hat{\beta}}\Psi_{\hat{\beta}} = (\psi^\alpha, -\bar{\phi}_{\dot{\alpha}})$ into a SU(2) doublet,

$$\Psi^{\hat{\alpha}}_i = (\Psi^\alpha_i, -\bar{\Psi}_{\dot{\alpha} i})\, , \qquad (\Psi^\alpha_i)^* = \bar{\Psi}^{\dot{\alpha} i}\, , \qquad i = \underline{1}, \underline{2}\, , \tag{A.5}$$

with $\Psi^\alpha_{\underline{1}} = \phi^\alpha$ and $\Psi^\alpha_{\underline{2}} = \psi^\alpha$. It is understood that the SU(2) indices are raised and lowered by ε^{ij} and ε_{ij}, $\varepsilon^{\underline{12}} = \varepsilon_{\underline{21}} = 1$, in the standard fashion: $\Psi^{\hat\alpha i} = \varepsilon^{ij}\Psi^{\hat\alpha}_j$. The Dirac spinor $\Psi^i = (\Psi^i_{\hat\alpha})$ satisfies the pseudo-Majorana condition $\bar\Psi_i{}^{\rm T} = C\,\Psi_i$. This will be concisely represented as

$$(\Psi^i_{\hat\alpha})^* = \Psi^{\hat\alpha}_i\ . \tag{A.6}$$

With the definition $\Sigma_{\hat m\hat n} = -\Sigma_{\hat n\hat m} = -\frac{1}{4}[\Gamma_{\hat m},\Gamma_{\hat n}]$, the matrices $\{\mathbb{1},\Gamma_{\hat m},\Sigma_{\hat m\hat n}\}$ form a basis in the space of 4×4 matrices. The matrices $\varepsilon_{\hat\alpha\hat\beta}$ and $(\Gamma_{\hat m})_{\hat\alpha\hat\beta}$ are antisymmetric, $\varepsilon^{\hat\alpha\hat\beta}\,(\Gamma_{\hat m})_{\hat\alpha\hat\beta} = 0$, while the matrices $(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}$ are symmetric. Note that any 4×4 matrix $\mathrm{B} = (\mathrm{B}_{\hat\alpha}{}^{\hat\beta})$ can be represented in the form:

$$\begin{aligned}
\mathrm{B} &= B\,\mathbb{1} + B^{\hat m}\,\Gamma_{\hat m} + \frac{1}{2}B^{\hat m\hat n}\,\Sigma_{\hat m\hat n}\,,\\
B &= \frac{1}{4}\mathrm{tr}\,\mathrm{B}\,,\quad B^{\hat m} = -\frac{1}{4}\mathrm{tr}\big(\Gamma^{\hat m}\,\mathrm{B}\big)\,,\quad B^{\hat m\hat n} = -\mathrm{tr}\big(\Sigma^{\hat m\hat n}\,\mathrm{B}\big)\ .
\end{aligned} \tag{A.7}$$

Given a 5-vector $V^{\hat m}$ and an antisymmetric tensor $F^{\hat m\hat n} = -F^{\hat n\hat m}$, we can equivalently represent them as the bi-spinors $V = V^{\hat m}\,\Gamma_{\hat m}$ and $F = \frac{1}{2}F^{\hat m\hat n}\,\Sigma_{\hat m\hat n}$ with the following symmetry properties

$$V_{\hat\alpha\hat\beta} = -V_{\hat\beta\hat\alpha}\,,\qquad \varepsilon^{\hat\alpha\hat\beta}\,V_{\hat\alpha\hat\beta} = 0\,,\qquad F_{\hat\alpha\hat\beta} = F_{\hat\beta\hat\alpha}\ . \tag{A.8}$$

The two equivalent descriptions $V_{\hat m} \leftrightarrow V_{\hat\alpha\hat\beta}$ and $F_{\hat m\hat n} \leftrightarrow F_{\hat\alpha\hat\beta}$ are explicitly described as follows:

$$\begin{aligned}
V_{\hat\alpha\hat\beta} &= V^{\hat m}\,(\Gamma_{\hat m})_{\hat\alpha\hat\beta}\,, & V_{\hat m} &= -\frac{1}{4}\,(\Gamma_{\hat m})^{\hat\alpha\hat\beta}\,V_{\hat\alpha\hat\beta}\,,\\
F_{\hat\alpha\hat\beta} &= \frac{1}{2}F^{\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}\,, & F_{\hat m\hat n} &= (\Sigma_{\hat m\hat n})^{\hat\alpha\hat\beta}\,F_{\hat\alpha\hat\beta}\ .
\end{aligned} \tag{A.9}$$

More generally, it holds

$$(\Gamma^{\hat m})_{\hat\alpha\hat\beta}(\Gamma^{\hat n})_{\hat\gamma\hat\delta}F_{\hat m\hat n} = 2\Big(\varepsilon_{\hat\alpha\hat\gamma}F_{\hat\beta\hat\delta} + \varepsilon_{\hat\beta\hat\delta}F_{\hat\alpha\hat\gamma} - \varepsilon_{\hat\alpha\hat\delta}F_{\hat\beta\hat\gamma} - \varepsilon_{\hat\beta\hat\gamma}F_{\hat\alpha\hat\delta}\Big)\ . \tag{A.10}$$

These results follow from the identities

$$\begin{aligned}
\varepsilon_{\hat\alpha\hat\beta\hat\gamma\hat\delta} &= \varepsilon_{\hat\alpha\hat\beta}\,\varepsilon_{\hat\gamma\hat\delta} + \varepsilon_{\hat\alpha\hat\gamma}\,\varepsilon_{\hat\delta\hat\beta} + \varepsilon_{\hat\alpha\hat\delta}\,\varepsilon_{\hat\beta\hat\gamma}\,,\\
(\Gamma^{\hat m})_{\hat\alpha\hat\beta}\,(\Gamma_{\hat m})_{\hat\gamma\hat\delta} &= \varepsilon_{\hat\alpha\hat\beta}\,\varepsilon_{\hat\gamma\hat\delta} - 2\varepsilon_{\hat\alpha\hat\gamma}\,\varepsilon_{\hat\beta\hat\delta} + 2\varepsilon_{\hat\alpha\hat\delta}\,\varepsilon_{\hat\beta\hat\gamma}\,,
\end{aligned} \tag{A.11}$$

which imply

$$\varepsilon_{\hat\alpha\hat\beta\hat\gamma\hat\delta} = \frac{1}{2}\,(\Gamma^{\hat m})_{\hat\alpha\hat\beta}\,(\Gamma_{\hat m})_{\hat\gamma\hat\delta} + \frac{1}{2}\,\varepsilon_{\hat\alpha\hat\beta}\,\varepsilon_{\hat\gamma\hat\delta}\,, \tag{A.12}$$

with $\varepsilon_{\hat\alpha\hat\beta\hat\gamma\hat\delta}$ the completely antisymmetric fourth-rank tensor. Complex conjugation gives

$$(\varepsilon_{\hat\alpha\hat\beta})^* = -\varepsilon^{\hat\alpha\hat\beta}\,,\qquad (V_{\hat\alpha\hat\beta})^* = V^{\hat\alpha\hat\beta}\,,\qquad (F_{\hat\alpha\hat\beta})^* = F^{\hat\alpha\hat\beta}\,, \tag{A.13}$$

provided $V^{\hat m}$ and $F^{\hat m\hat n}$ are real.

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We often make use of the completely antisymmetric tensor $\varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}$ that is normalized as $\varepsilon_{01235} = -\varepsilon^{01235} = 1$ and possesses the property

$$\varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{m}}\varepsilon_{\hat{m}\hat{a}'\hat{b}'\hat{c}'\hat{d}'} = -24\delta^{[\hat{a}}_{[\hat{a}'}\delta^{\hat{b}}_{\hat{b}'}\delta^{\hat{c}}_{\hat{c}'}\delta^{\hat{d}]}_{\hat{d}']} = -24\delta^{[\hat{a}}_{\hat{a}'}\delta^{\hat{b}}_{\hat{b}'}\delta^{\hat{c}}_{\hat{c}'}\delta^{\hat{d}]}_{\hat{d}'} = -24\delta^{\hat{a}}_{[\hat{a}'}\delta^{\hat{b}}_{\hat{b}'}\delta^{\hat{c}}_{\hat{c}'}\delta^{\hat{d}}_{\hat{d}']} \ . \tag{A.14}$$

It is useful to tabulate the products of several gamma-matrices (A.2). Making use of (A.7) gives

$$\Gamma^{\hat{a}}\Gamma^{\hat{b}} = -\eta^{\hat{a}\hat{b}}\mathbb{1} - 2\Sigma^{\hat{a}\hat{b}} \,, \tag{A.15a}$$

$$\Gamma^{\hat{a}}\Gamma^{\hat{b}}\Gamma^{\hat{c}} = (-\eta^{\hat{a}\hat{b}}\eta^{\hat{c}\hat{d}} + \eta^{\hat{c}\hat{a}}\eta^{\hat{b}\hat{d}} - \eta^{\hat{b}\hat{c}}\eta^{\hat{a}\hat{d}})\,\Gamma_{\hat{d}} + \varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}\,\Sigma_{\hat{d}\hat{e}} \,, \tag{A.15b}$$

$$\begin{aligned}\Gamma^{\hat{a}}\Gamma^{\hat{b}}\Gamma^{\hat{c}}\Gamma^{\hat{d}} &= (\eta^{\hat{a}\hat{b}}\eta^{\hat{c}\hat{d}} - \eta^{\hat{a}\hat{c}}\eta^{\hat{b}\hat{d}} + \eta^{\hat{a}\hat{d}}\eta^{\hat{b}\hat{c}})\mathbb{1} - \varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}\,\Gamma_{\hat{e}} + 2\eta^{\hat{a}\hat{b}}\,\Sigma^{\hat{c}\hat{d}} \\ &\quad -2\eta^{\hat{a}\hat{c}}\,\Sigma^{\hat{b}\hat{d}} + 2\eta^{\hat{b}\hat{c}}\,\Sigma^{\hat{a}\hat{d}} + 2\eta^{\hat{d}\hat{c}}\,\Sigma^{\hat{a}\hat{b}} - 2\eta^{\hat{d}\hat{b}}\,\Sigma^{\hat{a}\hat{c}} + 2\eta^{\hat{d}\hat{a}}\Sigma^{\hat{b}\hat{c}} \,,\end{aligned} \tag{A.15c}$$

$$\begin{aligned}\Gamma^{\hat{a}}\Gamma^{\hat{b}}\Gamma^{\hat{c}}\Gamma^{\hat{d}}\Gamma^{\hat{e}} &= \varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}\mathbb{1} + \Gamma^{\hat{a}}(\eta^{\hat{b}\hat{c}}\eta^{\hat{d}\hat{e}} - \eta^{\hat{b}\hat{d}}\eta^{\hat{c}\hat{e}} + \eta^{\hat{c}\hat{d}}\eta^{\hat{b}\hat{e}}) \\ &\quad +\Gamma^{\hat{b}}(-\eta^{\hat{c}\hat{d}}\eta^{\hat{e}\hat{a}} + \eta^{\hat{c}\hat{e}}\eta^{\hat{d}\hat{a}} - \eta^{\hat{d}\hat{e}}\eta^{\hat{c}\hat{a}}) + \Gamma^{\hat{c}}(\eta^{\hat{d}\hat{e}}\eta^{\hat{a}\hat{b}} - \eta^{\hat{d}\hat{a}}\eta^{\hat{e}\hat{b}} + \eta^{\hat{e}\hat{a}}\eta^{\hat{d}\hat{b}}) \\ &\quad +\Gamma^{\hat{d}}(-\eta^{\hat{e}\hat{a}}\eta^{\hat{b}\hat{c}} + \eta^{\hat{e}\hat{b}}\eta^{\hat{a}\hat{c}} - \eta^{\hat{a}\hat{b}}\eta^{\hat{e}\hat{c}}) + \Gamma^{\hat{e}}(\eta^{\hat{a}\hat{b}}\eta^{\hat{c}\hat{d}} - \eta^{\hat{c}\hat{a}}\eta^{\hat{b}\hat{d}} + \eta^{\hat{b}\hat{c}}\eta^{\hat{a}\hat{d}}) \\ &\quad +2\varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{m}}\,\Sigma_{\hat{m}}{}^{\hat{e}} - \eta^{\hat{a}\hat{b}}\varepsilon^{\hat{c}\hat{d}\hat{e}\hat{m}\hat{n}}\,\Sigma_{\hat{m}\hat{n}} + \eta^{\hat{c}\hat{a}}\varepsilon^{\hat{b}\hat{d}\hat{e}\hat{m}\hat{n}}\,\Sigma_{\hat{m}\hat{n}} - \eta^{\hat{b}\hat{c}}\varepsilon^{\hat{a}\hat{d}\hat{e}\hat{m}\hat{n}}\,\Sigma_{\hat{m}\hat{n}} \\ &\quad -\eta^{\hat{d}\hat{a}}\varepsilon^{\hat{b}\hat{c}\hat{e}\hat{m}\hat{n}}\,\Sigma_{\hat{m}\hat{n}} + \eta^{\hat{d}\hat{b}}\varepsilon^{\hat{a}\hat{c}\hat{e}\hat{m}\hat{n}}\,\Sigma_{\hat{m}\hat{n}} - \eta^{\hat{d}\hat{c}}\varepsilon^{\hat{a}\hat{b}\hat{e}\hat{m}\hat{n}}\,\Sigma_{\hat{m}\hat{n}} \ .\end{aligned} \tag{A.15d}$$

In conclusion, we give a useful relation often used in the paper. It is

$$\begin{aligned}\varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}(\Gamma^{\hat{c}})_{\hat{\alpha}\hat{\beta}}(\Sigma^{\hat{d}\hat{e}})_{\hat{\gamma}\hat{\delta}} &= 2\varepsilon_{\hat{\alpha}\hat{\beta}}(\Sigma_{\hat{a}\hat{b}})_{\hat{\gamma}\hat{\delta}} + 2\varepsilon_{\hat{\gamma}\hat{\alpha}}(\Sigma_{\hat{a}\hat{b}})_{\hat{\beta}\hat{\delta}} + 2\varepsilon_{\hat{\delta}\hat{\alpha}}(\Sigma_{\hat{a}\hat{b}})_{\hat{\beta}\hat{\gamma}} \\ &\quad -2\varepsilon_{\hat{\gamma}\hat{\beta}}(\Sigma_{\hat{a}\hat{b}})_{\hat{\alpha}\hat{\delta}} - 2\varepsilon_{\hat{\delta}\hat{\beta}}(\Sigma_{\hat{a}\hat{b}})_{\hat{\alpha}\hat{\gamma}} \ .\end{aligned} \tag{A.16}$$

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